Section 5.4 : Eigenvalues and Eigenvectors

Plan: * Examples : Computing Eigenpairs

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* Examples

Example: [Find the eigenpairs of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.]

Sol:

(a) Eigenvalues

$$p(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \det\left(\begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix} \right)$$

$$= \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = (\lambda - 1)^2 - 9 .$$

$$p(\lambda) = 0 \Rightarrow (\lambda_\pm - 1)^2 - 9 = 0 \Rightarrow \lambda_\pm - 1 = \pm 3$$

$$\Rightarrow \lambda_\pm = 1 \pm 3 \Rightarrow \boxed{\begin{array}{l} \lambda_+ = 4 \\ \hline \lambda_- = -2 \end{array}}$$

Now we compute the associated eigenvectors.

We start with $\underline{\lambda = 4}$

$\underline{\lambda_+ = 4}$; $A - \underline{4}I = \begin{bmatrix} 1-4 & 3 \\ 3 & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$

$$\begin{vmatrix} -3 & 3 \\ 3 & -3 \end{vmatrix} = 9 - 9 = 0$$

$$(A - 4I)\vec{V} = \vec{0} \Rightarrow \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \quad -3V_1 + 3V_2 = 0 \rightarrow -V_1 + V_2 = 0 \Rightarrow \boxed{V_1 = V_2}$$

$$3V_1 - 3V_2 = 0 \rightarrow V_1 - V_2 = 0$$

$$\vec{V}_+ = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_1 \end{bmatrix} = V_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \text{choose } V_1 = 1 \Rightarrow \boxed{\vec{V}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\boxed{\lambda_+ = 4}$$

Now $\boxed{\lambda = -2}$

$$A - \lambda I = A + 2I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$$(A - \lambda I)\vec{V} = \vec{0} \Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{array}{l} 3V_1 + 3V_2 = 0 \\ 3V_1 + 3V_2 = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad V_1 + V_2 = 0 \Rightarrow \boxed{V_1 = -V_2}$$

$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -V_2 \\ V_2 \end{bmatrix} = V_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \text{Choose } V_2 = 1 \Rightarrow \boxed{\vec{V}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$\boxed{\lambda_- = -2}$$

Example: Find the eigenpairs of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ where $A: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a Real function

Sol:

$$p(\lambda) = \begin{vmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

$$p(\lambda) = 0 \quad \Rightarrow \quad (\lambda_\pm)^2 = -1$$

Since $A: \mathbb{R}^2 \to \mathbb{R}^2$ (Real function) $\Rightarrow$ No eigenvalues
No eigenvectors

Example: Find the eigenpairs of $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ where $A: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ is a complex function

Sol:

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 1 = (\lambda-2)^2 + 1$$

$$p(\lambda) = 0 \quad \Rightarrow \quad (\lambda-2)^2 + 1 = 0 \quad \Rightarrow \quad \lambda_\pm - 2 = \pm\sqrt{-1}$$

$$\Rightarrow \quad \boxed{\lambda_\pm = 2 \pm i}$$

complex eigenvalues
(matrix A is a complex-valued function.)

Let $\lambda_+ = 2+i$,

$$A - \lambda_+ I = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 2+i & 0 \\ 0 & 2+i \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

$$(A - \lambda_+ I)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} -iv_1 + v_2 = 0 \\ -v_1 - iv_2 = 0 \end{array}\right\}$$ These eqs. are proportional to each other, since

$$(-v_1 - iv_2)i = 0i \Rightarrow -iv_1 + v_2 = 0$$

so, we only have the eq. $-iv_1 + v_2 = 0$ $\Rightarrow$

$$\Rightarrow \boxed{v_2 = iv_1}$$

Therefore, $\vec{v}_+ = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ iv_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ i \end{bmatrix}$.

choose $v_1 = 1$, then $\boxed{\vec{v}_+ = \begin{bmatrix} 1 \\ i \end{bmatrix}}$ $\boxed{\lambda_+ = 2+i}$

Now, since $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ has real coefficients,

$$A\vec{v}_+ = \lambda_+ \vec{v}_+ \Rightarrow \overline{A\vec{v}_+} = \overline{\lambda_+ \vec{v}_+}$$ ← complex conjugate.

$$\Rightarrow \overline{A}\,\overline{\vec{v}_+} = \overline{\lambda_+}\,\overline{\vec{v}_+} \; ; \text{ but } \overline{A} = A, \; \overline{\lambda_+} = 2-i = \lambda_-$$

So $A\overline{\vec{v}_+} = \lambda_- \overline{\vec{v}_+}$ $\Rightarrow$ $\boxed{\vec{v}_- = \overline{\vec{v}_+}}$

Therefore $\boxed{\vec{v}_- = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$ $\boxed{\lambda_- = 2-i}$

__Example__ : $\left[\begin{array}{l} \text{Find eigenpairs of } A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \\ A: \mathbb{R}^2 \to \mathbb{R}^2 \end{array}\right]$

__Sol.__

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 0 = (\lambda-2)^2$$

$$p(\lambda) = 0 \Rightarrow (\lambda_{\pm} - 2)^2 = 0 \Rightarrow \lambda_{\pm} = 2 \Rightarrow$$

$$\Rightarrow \boxed{\lambda_0 = \lambda_{\pm} = 2}$$

Eigenvectors: $(A - 2I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 2-2 & 1 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \boxed{v_2 = 0} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

choose $v_1 = 1 \Rightarrow \boxed{\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_0 = 2}$

This matrix has only one eigenpair.

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