Section 5.3 : Matrix Algebra

Plan:

* The Need of Matrix Algebra ✓
* Matrix Operations (2x2) ✓

In Class

* Determinant and Inverse of 2x2 Matrices ✓
* Equations for Matrices ✓
* Determinant of 3x3 Matrices ✓

This Video

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* Inverse of a Matrix

Def: The $n \times n$ identity matrix

$$I_n = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

Example: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Remark:

$$\begin{cases} I_n \vec{v} = \vec{v} & \text{all } n\text{-vector} \\ I_n A = A \quad A I_n = A & \text{all } n \times n \text{ matrices} \end{cases}$$

Def: An $n \times n$ matrix A is invertible iff there is an $n \times n$ matrix A^{-1} s.t.

$$A^{-1} A = I_n \qquad A A^{-1} = I_n$$

Remark: A 2x2 Algebraic Linear System (ALS)

$$\begin{bmatrix} a_{11} x_1 + a_{12} x_2 = b_1 \\ a_{21} x_1 + a_{22} x_2 = b_2 \end{bmatrix}$$

unknowns x_1, x_2

Example : $\left[\begin{array}{l} 2x_1 + 2x_2 = 1 \\ x_1 + 3x_2 = 2 \end{array}\right]$ (Substit.)

unknowns

Matrix Notation

$\boxed{A\vec{x} = \vec{b}}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

If A is invertible, with inverse A^{-1},

$\underbrace{A^{-1}A}_{=I_2}\vec{x} = A^{-1}\vec{b}$ $\Rightarrow$ $I_2\vec{x} = A^{-1}\vec{b}$

$\Rightarrow$ $\boxed{\vec{x} = A^{-1}\vec{b}}$

A^{-1} is useful to find sols. of ALS.

* Formula for Inverse of a 2x2 matrix

Example : $\left[\begin{array}{l} \text{Solve } A\vec{x} = b \text{ with} \\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{array}\right]$

Sol

$A\vec{x} = \vec{b} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\begin{array}{rl} & d\,(ax_1 + bx_2 = b_1) \\ - & b\,(cx_1 + dx_2 = b_2) \\ \hline \end{array}$$

$$adx_1 \cancel{+ dbx_2} - bcx_1 \cancel{- bdx_2} = db_1 - bb_2$$

$$\underbrace{(ad - bc)}_{\neq 0}x_1 = db_1 - bb_2$$

$\boxed{\Delta = ad - bc}$ assume $\Delta \neq 0$

$$\boxed{x_1 = \frac{1}{\Delta}\left(d\, b_1 - b\, b_2\right)}$$

$$c\,(a x_1 + b x_2 = b_1)$$
$$-\; a\,(c x_1 + d x_2 = b_2)$$

$$\cancel{ca x_1} + cb\, x_2 - \cancel{ac x_1} - ad\, x_2 = c b_1 - a b_2$$

$$\underbrace{(cb - ad)}_{\neq 0}\, x_2 = c\, b_1 - a\, b_2$$

$$-\Delta\, x_2 = c\, b_1 - a\, b_2$$

$$\boxed{x_2 = \frac{1}{\Delta}\left(-c\, b_1 + a\, b_2\right)}$$

$$\boxed{\begin{aligned} x_1 &= \frac{1}{\Delta}\left(d\, b_1 - b\, b_2\right) \\ x_2 &= \frac{1}{\Delta}\left(-c\, b_1 + a\, b_2\right) \end{aligned}}$$

we want

$$\vec{x} = A^{-1}\, \vec{b}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} d\, b_1 & -b\, b_2 \\ -c\, b_1 & +a\, b_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\vec{x} = \left(A^{-1}\right)\vec{b}$$

$$\boxed{A^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$\Delta \neq 0$

$\Delta = ad - bc$

Def : The determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\underbrace{\Delta = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}}_{\text{names}} = \underline{ad - bc}$$

Thrm (2x2) : A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible

iff $\underline{\det(A) \neq 0}$

and in this case

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

check :

$$A^{-1}A = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{(ad-bc)} \begin{bmatrix} (ad-bc) & (db-bd) \\ (-ac+ac) & (-bc+ad) \end{bmatrix}$$

$$= \frac{1}{(ad-bc)} \begin{bmatrix} (ad-bc) & 0 \\ 0 & (ad-bc) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \quad \checkmark$$

$AA^{-1} = I_2 \quad \leftarrow \quad$ exercise

* **Example** : Find A^{-1} for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

sol

Is A invertible ?

$$\det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = \underline{-2} \neq 0 \quad \checkmark$$

$$\boxed{A^{-1} = \frac{1}{(-2)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}} \quad \left(= \underline{\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}} \right)$$

$$A A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \frac{1}{(-2)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{(-2)} \begin{bmatrix} (4-6) & (-2+2) \\ (12-12) & (-6+4) \end{bmatrix}$$

$$= \frac{1}{(-2)} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 .$$

$\triangle$

<u>Example</u> : $\left[\text{Solve} \quad \begin{matrix} 2x_1 + 2x_2 = 1 \\ x_1 + 3x_2 = 2 \end{matrix} \right]$

<u>sol</u>

$$A\vec{x} = \vec{b} \quad , \quad A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Is A invertible ? $\quad \det(A) = 6 - 2 = 4 \neq 0 \quad \checkmark$

$$\boxed{\vec{x} = \underline{A^{-1}}\vec{b}}$$

$$\vec{x} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3-4 \\ -1+4 \end{bmatrix}$$

$$\boxed{\vec{x} = \frac{1}{4} \begin{bmatrix} -1 \\ 3 \end{bmatrix}} = \begin{bmatrix} -1/4 \\ 3/4 \end{bmatrix} .$$

$\triangle$

Example: [Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ is **Not** invert.]

sol.

$$\det(A) = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 9 - 9 = 0$$

Remark: $A = \left[\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \end{bmatrix} \right]$

$$\downarrow \quad \downarrow$$

$$\vec{a}_1 \quad \vec{a}_2 \quad \Rightarrow \quad \vec{a}_2 = \underline{3}\, \vec{a}_1$$

* Equations for Matrices

Example: [Find a matrix X s.t. $\underline{AXB = C}$ where
$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$]

sol

Since $\left.\begin{matrix} A & X & B \\ 2\times 2 & m\times n & 2\times 2 \end{matrix}\right\} \Rightarrow \underline{X \text{ is } 2\times 2}$

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} (\text{———}) = 1 & (\text{———}) = 2 \\ (\text{———}) = 2 & (\text{———}) = 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$\underline{x_{11}, x_{12}, x_{21}, x_{22}}$

$$\left.\begin{array}{l}(\text{———}) = 1\\ (\text{———}) = 2\\ (\text{———}) = 2\\ (\text{———}) = 1\end{array}\right\}$$ 4×4 LAS

—o—

$AXB = C$ where

$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Is A invertible? $\det(A) = 1 - 6 = -5 \neq 0$ ✓

Is B invertible? $\det(B) = 4 - 1 = 3 \neq 0$ ✓

$$\underbrace{A^{-1} A}_{I_2} X B = A^{-1} C$$

$$I_2 X B = A^{-1} C \qquad I_2 X = X$$

$$B^{-1} X B = \underline{\underline{B^{-1} A^{-1} C}} \qquad B^{-1} X \neq X B^{-1}$$

$$X \underbrace{B B^{-1}}_{= I_2} = A^{-1} C B^{-1}$$

$$\boxed{X = A^{-1} C B^{-1}}$$

$$A^{-1} A X B B^{-1} = A^{-1} C B^{-1}$$

$$X = \underline{\underline{A^{-1} C B^{-1}}}$$ ←

$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$X = \frac{1}{(-5)} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \left(\frac{1}{3}\right) \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \left(\frac{1}{-15}\right) \begin{bmatrix} -5 & -1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \left(\frac{1}{-15}\right) \begin{bmatrix} -9 & 3 \\ 3 & -6 \end{bmatrix}$$

$$\boxed{X = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}}$$

✓

* <u>Det of 3x3 Matrices</u>

<u>Remarks</u>: - <u>det(A)</u>, <u>A 2x2</u>,
↳ number determines whether A is invertible or not.

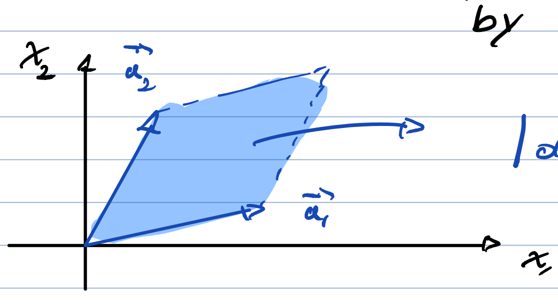
- can we find det(A), A nxn with the same property?

- Yes; example for <u>3x3</u>.

- The generalization uses the <u>geometrical meaning</u> of det for 2x2.

<u>Remark</u>: * $A = [\vec{a}_1, \vec{a}_2]$, 2x2

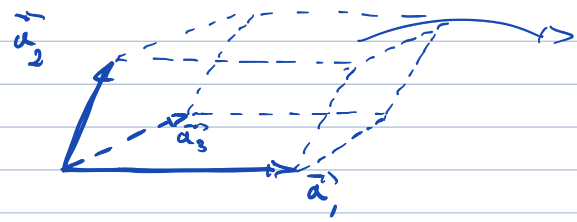
$|\det(A)|$ = Area of the parallelogram formed by $\vec{a}_1$, $\vec{a}_2$.



$|\det(A)|$

* $A = [\vec{a}_1, \vec{a}_2, \vec{a}_3]$, 3×3.

we define $\det(A)$ so that $|\det(A)|$ is volume of the parallelepiped formed by $\vec{a}_1, \vec{a}_2, \vec{a}_3$.



Def :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = + a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example : Find $\det(A)$, $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = +(1) \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - (3) \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= (1-2) - 3\,(\underbrace{2-3}_{-1}) - (4-3)$$

$$= -1 + 3 - 1$$

$$= 1$$