Section 5.2 : Vector Spaces

Plan: * Linear (In)Dependence

* Basis and Dimension

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Recall: In $V = \mathbb{R}^3$:

* $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ are l.d., then they lie on a plane or on a line.
  (They span a plane or a line.)

* $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ are l.i. iff they do Not lie on a plane.
  (They span the whole $\mathbb{R}^3$.)

Example: Determine l.d. or l.i.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$

Sol:

(a) $c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Solve for c_i.

(1)

$c_1 + 7c_2 + 3c_3 = 0$

✓ $2c_1 + 2c_2 + 2c_3 = 0$

✓ $3c_1 - 3c_2 + c_3 = 0 \Rightarrow c_3 = 3c_2 - 3c_1$ $\Big\} \Rightarrow$

$\Rightarrow 2c_1 + 2c_2 + 6c_2 - 6c_1 = 0$

$-4c_1 + 8c_2 = 0 \Rightarrow \boxed{c_1 = 2c_2}$

$c_3 = 3c_2 - 6c_2$

$\boxed{c_3 = -3c_2}$

$c_1 + 7c_2 + 3c_3 = 0$

$2c_2 + 7c_2 + 3(-3c_2) = 0$

$9c_2 - 9c_2 = 0$

$\boxed{0 c_2 = 0} \Rightarrow \boxed{c_2 : \text{free}}$

No condition on c_2 $\Rightarrow$ l. d.

Therefore, Eq (1) above is :

$$2c_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix} - 3c_2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

$$\boxed{\begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

l. d.

(b)

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\checkmark$ $c_1 + 0c_2 + 2c_3 = 0 \Rightarrow \boxed{c_1 = -2c_3}$

$\checkmark$ $0c_1 + c_2 + 3c_3 = 0 \Rightarrow \boxed{c_2 = -3c_3}$

$\Rightarrow \boxed{c_1 = \frac{2}{3} c_2}$

$\checkmark$ $c_1 + 2c_2 + 0c_3 = 0$

$\boxed{c_1 = -2c_2}$ $\quad$ $\boxed{c_1 = \frac{2}{3}c_2}$ $\Rightarrow \frac{2}{3}c_2 = -2c_2$

$8c_2 = 0 \Rightarrow$

$\Rightarrow c_2 = 0 \Rightarrow c_1 = 0$ and $c_3 = 0$

Therefore: The only solution is: $\boxed{c_1 = c_2 = c_3 = 0}$

$\boxed{\text{l. i.}}$

* Basis and Dimension

Remark: A basis of a vector (sub)space is the smallest set that spans that space.

Def: A set $S \subset V$ is a basis of a VS V iff
(a) $\text{Span}(S) = V$
(b) S is l.i.

Example: Determine if $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis of

$W = \text{Span}(S)$

Sol:

(a) ✓

(b) ? No, we saw that S is l.d.

$$\begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \boxed{W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}.}$$

So, is $S_0 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ a basis of W ?

(a) ✓, so check if S_0 is l.i.

$$0 = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \left. \begin{matrix} c_1 = -3c_2 \\ c_1 = -c_2 \end{matrix} \right\} \Rightarrow c_2 = 0, \; c_1 = 0 \quad ✓$$

So: $\boxed{S_0 \text{ is a basis of } W.}$ ■

Remark: A vector (sub)space has infinitely many bases.

Another basis of W above is:

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix} \right\}.$$

Another basis is:

$$S_2 = \left\{ \begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Another basis is:

$$S_3 = \left\{ \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

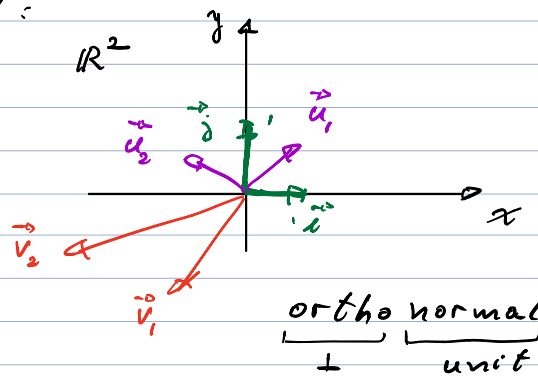
Thrm: [All bases of $W \subset \mathbb{R}^n$ have the same number of vectors]

Def: [The dimension of a (sub)space $W \subset \mathbb{R}^n$ is the number of vectors in any basis of W.]

Example: [
* S, S_0, S_1, S_2, S_3 are bases of W above.
* W has dimension $d = 2$.
* W is a plane in $\mathbb{R}^3$.
]

Example : [Find two bases of $\mathbb{R}^2$.] $\to$ 2 vectors, l.i.

Sol :



$S_0 = \{\vec{i}, \vec{j}\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

$S_1 = \{\vec{u}_1, \vec{u}_2\} = \left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\right\}$

$\underbrace{\text{ortho}}_{\perp}\underbrace{\text{normal}}_{\text{unit}}$
$\left\{\begin{array}{l} \vec{i} \perp \vec{j}, \quad |\vec{i}| = 1 = |\vec{j}|. \\ \vec{u}_1 \perp \vec{u}_2, \quad |\vec{u}_1| = 1, \quad |\vec{u}_2| = 1 \end{array}\right.$

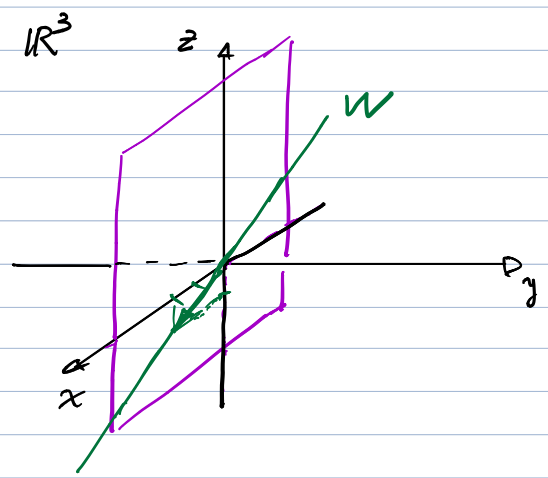
$S_2 = \left\{\vec{v}_1 = \begin{bmatrix}-5\\-7\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}-9\\-6\end{bmatrix}\right\}$

Example : [Find the dimension of

$W = \text{Span}\left\{\begin{bmatrix}2\\0\\-1\end{bmatrix}, \begin{bmatrix}-6\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\\frac{1}{2}\end{bmatrix}\right\} \subset \mathbb{R}^3$]

Sol.

$\left\{\begin{bmatrix}2\\0\\-1\end{bmatrix}, \begin{bmatrix}-6\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\\frac{1}{2}\end{bmatrix}\right\}$ l.d.



$-\frac{1}{2}\begin{bmatrix}2\\0\\-1\end{bmatrix} = \begin{bmatrix}-1\\0\\1/2\end{bmatrix}$

$\begin{bmatrix}-6\\0\\3\end{bmatrix} = -3\begin{bmatrix}2\\0\\1\end{bmatrix}$

$\text{Span}\left\{\begin{bmatrix}2\\0\\-1\end{bmatrix}, \begin{bmatrix}-6\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\1/2\end{bmatrix}\right\} =$

$= \text{Span}\left\{\begin{bmatrix}2\\0\\-1\end{bmatrix}, \begin{bmatrix}-6\\0\\3\end{bmatrix}\right\} =$

$= \text{Span}\left\{\underline{\begin{bmatrix}2\\0\\-1\end{bmatrix}}\right\}$

W is a line, $\dim W = 1$.

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