Section 2.3 Springs, Circuits, and Resonance

Plan:

* Springs and Circuits
* Forced Oscillations: $\begin{cases} \text{- Non-Resonant} \\ \text{- Resonant} \end{cases}$

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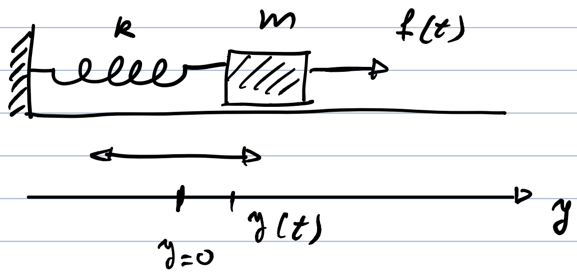
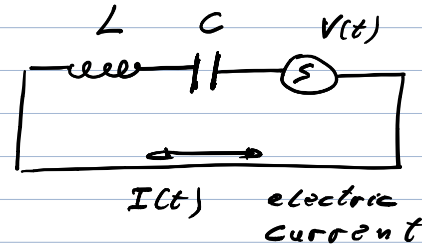
* Springs and Circuits

- Spring Oscillations.



(No Friction)

LC-Series Circuit



L: Inductor (coil)

C: Capacitor — conductor separated by insulator

$V(t)$: Voltage Source (power)

Mathematically Equivalent

Newton's Eq.

$$m y'' = \underbrace{-ky}_{\text{spring force}} + \underbrace{f(t)}_{\text{external force}}$$

$m=1, \quad k=25$

$$\boxed{y'' + 25y = f(t)}$$

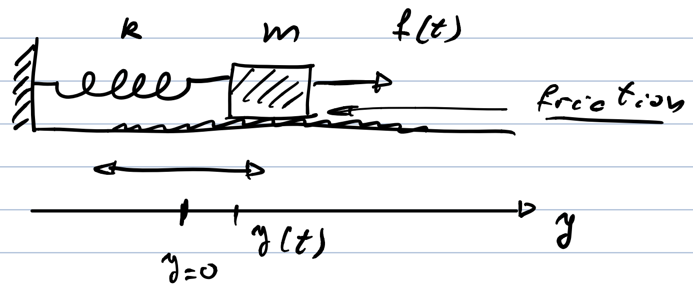
Kirchhoff's Eq.

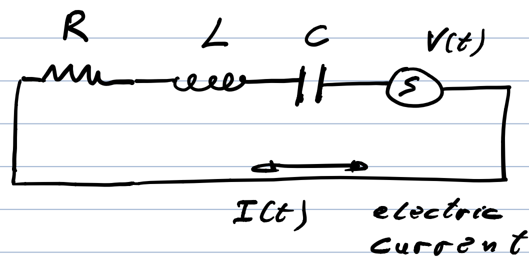
$$L I'' + \frac{1}{C} I = V'(t)$$

$L=1, \quad \frac{1}{C} = 25, \quad f(t) = V'(t)$

$$\boxed{y'' + 25y = f(t)}$$

Comment: Friction Terms.





With Friction $\alpha > 0$

$$\boxed{m y'' + \alpha y' + k y = f(t)}$$

Resistance $R > 0$

$$\boxed{L I'' + R I' + \frac{1}{C} I = V'(t)}$$

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Free oscillations : $f(t) = 0$

$$\boxed{y'' + 25y = 0} \Rightarrow r^2 + 25 = 0$$

$$r_\pm = \pm 5i$$

$$\boxed{y = c_1 \cos(5t) + c_2 \sin(5t)}$$

$\omega = 5$ natural freq.

Forced Oscillations (Non Resonant)

$$f(t) = \cos(\nu t) \qquad \nu \neq 5 = \omega$$

$$\boxed{y'' + 25y = \cos(\nu t)} \qquad \boxed{\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}}$$

$$\boxed{y(t) = \frac{1}{(5^2 - \nu^2)}\left(\cos(\nu t) - \cos(5t)\right)}$$

Forced Oscillations (Resonant)

$$f(t) = \cos(5t) \qquad \nu = 5 = \omega$$

$$\boxed{y'' + 25y = \cos(5t)} \qquad \boxed{\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}}$$

$$\boxed{y(t) = \frac{t}{10}\sin(5t)}$$

* Forced Oscillations: Non Resonant

Example: Solve the IVP

$$y'' + 25\, y = \cos(\nu t), \qquad \nu \neq 5$$

$$y(0) = 0, \qquad y'(0) = 0$$

Sol:

$$\underbrace{y'' + 25\, y}_{L(y)} = \cos(\nu t)$$

(1) Sols. Hom. Eq. $\quad y'' + 25\, y = 0 \quad \leftarrow \quad y = e^{rt} \Big\} \Rightarrow$

$\Rightarrow \quad r^2 + 25 = 0 \quad \Rightarrow \quad r_\pm = \pm\sqrt{-25}$

$$\boxed{r_\pm = \pm 5i}$$

Fund. Sols: $\quad y_+ = e^{5it}, \qquad y_- = e^{-5it}$

Real Fund Sols: $\quad \boxed{y_1 = \cos(5t)}, \qquad \boxed{y_2 = \sin(5t)}$

(2) Sol. Non-Hom. Eq.

$f(t) = \cos(\nu t) \quad \Rightarrow \quad \boxed{y_{p_1}(t) = k_1 \cos(\nu t) + k_2 \sin(\nu t)}$

Recall: $\nu \neq 5 \quad \Rightarrow \quad \underline{L(y_{p_1}) \neq 0} \quad \Rightarrow \quad y_{p_1}$ Good Guess.

(3) Compute k_1, k_2 from $L(y_{p_1}) = f$.

$$y'_{p_1} = k_1(-\nu)\sin(\nu t) + k_2 \nu \cos(\nu t)$$

$$y''_{p_1} = k_1(-\nu^2)\cos(\nu t) + k_2(-\nu^2)\sin(\nu t)$$

So:

$$\left(k_1(-\nu^2)\cos(\nu t) + k_2(-\nu^2)\sin(\nu t)\right) + 25\left(k_1\cos(\nu t) + k_2\sin(\nu t)\right) = \cos(\nu t)$$

$$\cos(\nu t)\left(-\nu^2 k_1 + 5^2 k_1 - 1\right) + \sin(\nu t)\left(-\nu^2 k_2 + 5^2 k_2\right) = 0$$

$$\cos(\nu t)\left(k_1(5^2-\nu^2) - 1\right) + \sin(\nu t)\, k_2(5^2-\nu^2) = 0$$

for all t.

$$\cos(\nu t)\,\alpha + \sin(\nu t)\,\beta = 0 \quad \text{all } t \Rightarrow \boxed{\begin{array}{l}\alpha = 0\\ \beta = 0\end{array}}$$

$t = \frac{\pi}{2}\frac{1}{\nu} \Rightarrow 0 + \beta = 0$ ✓

$t = 0 \Rightarrow \alpha + 0 = 0$ ✓

So: $\left[\begin{array}{ll} k_1(5^2-\nu^2) - 1 & = 0 \\ k_2(5^2-\nu^2) & = 0 \end{array}\right] \quad \nu \neq 5 \Bigg\} \Rightarrow \boxed{k_1 = \frac{1}{5^2-\nu^2}}$ $\boxed{k_2 = 0}$

$$\boxed{y_p(t) = \frac{1}{(5^2-\nu^2)}\cos(\nu t)}$$

Gen. Sol. $\boxed{y(t) = c_1\cos(5t) + c_2\sin(5t) + \frac{\cos(\nu t)}{(5^2-\nu^2)}}$

I.C. $\quad 0 = y(0) = c_1 + 0 + \frac{1}{5^2 - \nu^2} \quad \Rightarrow$

$$\Rightarrow \quad \boxed{c_1 = -\frac{1}{(5^2 - \nu^2)}}$$

$$y'(t) = -5c_1 \sin(5t) + 5c_2 \cos(5t) - \frac{\nu \sin(\nu t)}{(5^2 - \nu^2)}$$

I.C. $\quad 0 = y'(0) = 0 + 5c_2 - 0 \quad \Rightarrow$

$$\Rightarrow \quad \boxed{c_2 = 0}$$

$$y(t) = -\frac{1}{(5^2 - \nu^2)} \cos(5t) + \frac{1}{(5^2 - \nu^2)} \cos(\nu t)$$

$$\boxed{y(t) = \frac{1}{(5^2 - \nu^2)} \left(\cos(\nu t) - \cos(5t) \right)}$$

* Forced Oscillations: Resonant

Example: Solve the IVP

$$\left[\begin{array}{l} y'' + 25y = \cos(5t), \quad (\nu = 5) \\ y(0) = 0, \quad y'(0) = 0 \end{array}\right]$$

Sol: $$\boxed{\underbrace{y'' + 25y}_{L(y)} = \cos(5t)}$$

(1) Hom. Eq. $y'' + 25y = 0 \leftarrow y = e^{rt}$ } =>

=> $r^2 + 25 = 0$ => $\boxed{r_\pm = \pm 5i}$

$\boxed{y_1 = \cos(5t)}$, $\boxed{y_2 = \sin(5t)}$

(2) Non Hom. Eq.
First Guess.

$f = \cos(5t)$ => $y_{p_1} = k_1 \cos(5t) + k_2 \sin(5t)$

But now $L(y_{p_1}) = 0$. y_{p_1}: Wrong Guess.

(3) Second Guess.

$\boxed{y_{p_2} = k_1 t \cos(5t) + k_2 t \sin(5t)}$ => $L(y_{p_2}) \neq 0$.

$y'_{p_2} = k_1 \cos(5t) + k_2 \sin(5t) + k_1 t(-5) \sin(5t) + k_2 t(5) \cos(5t)$

$$\boxed{\begin{aligned} y_2'' = \; & 2 k_1 (-5) \sin(5t) + 2 k_2 (5) \cos(5t) \\ & - k_1 5^2 t \cos(5t) - k_2 5^2 t \sin(5t) \end{aligned}}$$

$$\Big(-10 k_1 \sin(5t) + 10 k_2 \cos(5t) - \cancel{k_1 5^2 t \cos(5t)} - \cancel{k_2 5^2 t \sin(5t)}\Big)$$
$$+ 25 \Big(\cancel{k_1 t \cos(5t)} + \cancel{k_2 t \sin(5t)} \Big) = \cos(5t)$$

$$-10 k_1 \sin(5t) + 10 k_2 \cos(5t) = \cos(5t)$$

$$(-10 k_1) \sin(5t) + (10 k_2 - 1) \cos(5t) = 0 \quad \text{all } t.$$

$\therefore \quad \boxed{k_1 = 0}$

$10 k_2 - 1 = 0 \quad \Rightarrow \quad \boxed{k_2 = \frac{1}{10}}$

$$\boxed{y_{p_2} = \frac{t}{10} \sin(5t)}$$

Gen. Sol. $\quad \boxed{y(t) = c_1 \cos(5t) + c_2 \sin(5t) + \frac{t}{10} \sin(5t)}$

I.C. $\quad 0 = y(0) = c_1 + 0 + 0. \quad \Rightarrow \quad \boxed{c_1 = 0}$

$$y'(t) = -5 c_1 \sin(5t) + 5 c_2 \cos(5t) + \frac{1}{10} \sin(5t) + \frac{5t}{10} \cos(5t)$$

$$0 = y'(0) = 0 + 5 c_2 + 0 + 0 \quad \Rightarrow \quad \boxed{c_2 = 0}$$

$$\boxed{y(t) = \frac{t}{10} \sin(5t)}$$