## * Properties of Homogeneous Eqs $(b=0)$

We focus on $y'' + a_1(t)\, y' + a_0(t)\, y = 0$

**Notation:** $L(y) = y'' + a_1(t)\, y' + a_0(t)\, y$

operator : a function on functions.

## (1) Linearity:

**Thrm:** $\left[\; L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2) \;\right]$.

**Proof**

$$\underline{L(c_1 y_1 + c_2 y_2)} = (c_1 y_1 + c_2 y_2)'' + a_1 (c_1 y_1 + c_2 y_2)' + a_0 (c_1 y_1 + c_2 y_2)$$

$$= c_1 y_1'' + c_2 y_2'' + a_1 c_1 y_1' + a_1 c_2 y_2' + a_0 c_1 y_1 + a_0 c_2 y_2$$

$$= c_1 \left(y_1'' + a_1 y_1' + a_0 y_1\right) + c_2 \left(y_2'' + a_1 y_2' + a_0 y_2\right)$$

$$= c_1 L(y_1) + c_2 L(y_2)$$

■

## (2) Superposition

**Thrm:** $\left[\; L(y_1) = 0,\ L(y_2) = 0 \implies L(c_1 y_1 + c_2 y_2) = 0 \;\right]$

**Proof:** $L(c_1 y_1 + c_2 y_2) = c_1 \underset{0}{\underset{\parallel}{L(y_1)}} + c_2 \underset{0}{\underset{\parallel}{L(y_2)}} = 0$.

■

## (3) General Solution

**Thrm:** [ If $y_1, y_2$, with $y_1 \neq c\, y_2$, are sol. of $L(y) = 0$, Then **every** solution $y$ of $L(y)=0$ is

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$$ ]

**Remark:** $y_1, y_2$ with $y_1 \neq c\, y_2$, $L(y_1)=0=L(y_2)$ are called **fundamental** solutions.

Example (Hw) (Superposition Property)

(a) $y_1$ is sol. of $y'' + a_1 y' + a_0 y = 0$. (1)

(b) $y_2$ is sol. of $\underbrace{y'' + a_1 y' + a_0 y}_{=L(y)} = \cos(2t)$ (2)

Then (T/F)

(1) $y_1 + y_2$ Solves (1). (F)

$$L(y_1+y_2) = L(y_1) + L(y_2) = 0 + \cos(2t)$$

(2) $y_1 + y_2$ Solves (2) (T)

(3) $2y_1$ Solves (1) (T)

$$L(2y_1) = 2\, L(y_1) = 2(0) = 0.$$

(4) $2y_2$ Solves (2) (F)

$$L(2y_2) = 2L(y_2) = 2\cos(2t) \neq \cos(2t)$$

**Remark:** If $L(y_h) = 0$, and $L(y_2) = b(t)$

Then $L(y_2(t) + c_1 y_h(t)) = b(t)$.

**Example:** [ Show $y_1 = e^{-3t}$, $y_2 = e^{2t}$ are fund. sols. of

$$\underbrace{y'' + y' - 6y}_{=L(y)} = 0 \quad \rightarrow \quad (1)$$ ]

Sol:

$$y_1 = e^{-3t} \neq c_0 e^{2t} = y_2$$

$$L(y_1) = L(e^{-3t}) = (e^{-3t})'' + (e^{-3t})' - 6e^{-3t}$$

$$= 9e^{-3t} - 3e^{-3t} - 6e^{-3t}$$

$$= (9-3-6)e^{-3t}$$

$$= 0$$

Show $L(y_2) = 0$.

Example : $Y = 3$ is sol of $\underbrace{y'' + y' - 6y}_{=L(y)} = -18.$ → (2)

Find infinitely many more sols.

Sol: $L(Y) = L(3) = 3'' + 3' - 6(3) = -18$ ✓

We know from precious Ex: $y_1 = e^{-3t}$, $y_2 = e^{2t}$ are solutions of

$L(y_1) = L(e^{-3t}) = 0$, $L(y_2) = L(e^{2t}) = 0.$

So $y = 3 + c_1 e^{-3t}$, satisfies

$$\begin{aligned} L(y) &= L(3 + c_1 e^{-3t}) \\ &= L(3) + c_1 L(e^{-3t}) \\ &= L(Y) + c_1 0 \\ &= -18 \end{aligned}$$ ✓

Also $y = 3 + c_2 e^{+2t}$

$$\begin{aligned} L(y) &= L(3 + c_2 e^{2t}) \\ &= L(3) + c_2 L(e^{2t}) \\ &= -18 + c_2 0 \\ &= -18 \end{aligned}$$

$y(t) = 3 + c_1 e^{-3t} + c_2 e^{2t}$

satisfies $L(y) = -18$

— o —

__Example__ : Find the maximum domain where the sol. of IVP is certain to exist,

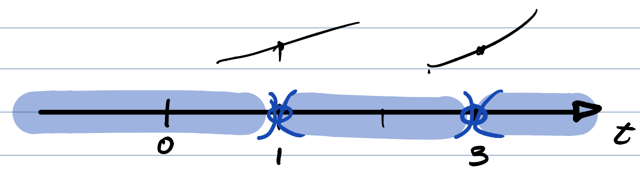
$$(t-1)\, y'' - 3t\, y' + \frac{4(t-1)}{(t-3)}\, y = t(t-1)$$

$$y(2) = 1, \qquad y'(2) = 0$$

__Sol:__

$$y'' \underbrace{- \frac{3t}{(t-1)}}_{= a_1} y' + \underbrace{\frac{4}{(t-3)}}_{= a_0} y = \underbrace{t}_{b}$$

The eq is defined on




$(-\infty, 1) \cup (1, 3) \cup (3, \infty)$.

The solution is defined (for certain) on

$(-\infty, 1)$ or $(1, 3)$ or $(3, \infty)$.

The IC is at $t_0 = 2 \in (1, 3)$.

So $y(t)$ is certain to exist on

$$\boxed{D_y = (1, 3)}$$

∎