Example: Use the Picard iteration to solve

$$y' = 2y + 3 \quad , \quad y(0) = 1 .$$

Sol: Transform the differential eq. into an integral eq.

$$\int_0^t y'(s)\, ds = \int_0^t (2\, y(s) + 3)\, ds$$

$$y(t) - y(0) = 2 \left(\int_0^t y(s)\, ds \right) + 3t$$

$$\boxed{y(t) = 1 + 2 \left(\int_0^t y(s)\, ds \right) + 3t .}$$

Now define the Picard iteration:

$$y_0 = y(0) = 1$$

$$y_{n+1}(t) = 1 + 2 \left(\int_0^t y_n(s)\, ds \right) + 3t \quad , \qquad n \geq 0 .$$

Let's compute a few terms:

$n=0$: $\quad y_1(t) = 1 + 2 \left(\int_0^t y_0\, ds \right) + 3t \quad ; \quad y_0 = 1$

$$y_1(t) = 1 + 2t + 3t \quad \Rightarrow \quad \boxed{y_1(t) = 1 + 5t}$$

$n=1$: $\quad y_2(t) = 1 + 2 \left(\int_0^t y_1(s)\, ds \right) + 3t$

$$= 1 + 2 \left(\int_0^t (1 + 5s)\, ds \right) + 3t$$

$$= 1 + 2 \left(t + \frac{5}{2} t^2 \right) + 3t$$

$$\boxed{y_2(t) = 1 + 5t + 5t^2}$$

$n=2$: $\quad y_3(t) = 1 + 2 \left(\int_0^t y_2(s)\, ds \right) + 3t$

$$= 1 + 2 \int_0^t (1 + 5s + 5s^2)\, ds + 3t$$

$$= 1 + 2 \left(t + \frac{5}{2} t^2 + \frac{5}{3} t^3 \right) + 3t$$

$$\boxed{y_3(t) = 1 + 5t + 5t^2 + \frac{10}{3} t^3}$$

Finding the general term.

We are solving a linear eq. We know that <u>exponentials</u> are solutions.

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots}$$

<u>Rewrite</u> y_3.

$$y_3(t) = 1 + 5t + 5t^2 + 5(2)\frac{t^3}{3}$$

We need the factorials $n!$ on terms with powers t^n

I multiplied by $\frac{2}{2}$ to make a $2!$

I also multiplied by $\frac{2}{2}$ to create a $3!$

$$y_3(t) = 1 + 5\frac{t}{1!} + 5(2)\frac{t^2}{2!} + 5(2^2)\frac{t^3}{3!}$$

5 is a common factor that is independent of the power n in t^n.

$$y_3(t) = 1 + 5\left(\frac{t}{1!} + \frac{2t^2}{2!} + \frac{2^2 t^3}{3!}\right)$$

The factor 2 increases with $n-1$, when t is t^n.
So multiply and divide by 2.

$$y_3(t) = 1 + 5\left(\frac{2}{2}\right)\left(\frac{t}{1!} + \frac{2t^2}{2!} + \frac{2^2t^3}{3!}\right)$$

$$\boxed{y_3(t) = 1 + \frac{5}{2}\left(\frac{(2t)}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!}\right)}$$

So the general term is

$$y_N(t) = 1 + \frac{5}{2}\left(\sum_{n=1}^{N} \frac{(2t)^n}{n!}\right)$$

and the limit is:

$$y(t) = \lim_{N\to\infty} y_N(t) = 1 + \frac{5}{2} \sum_{n=1}^{\infty} \frac{(2t)^n}{n!}$$

Recall that: $\quad e^{2t} = \sum_{n=0}^{\infty} \frac{(2t)^n}{n!}$

$$= 1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(2t)^n}{n!} \quad \Rightarrow$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{(2t)^n}{n!} = e^{2t} - 1 .$$

Therefore: $\quad y(t) = 1 + \frac{5}{2}\left(e^{2t} - 1\right) = \frac{5}{2} e^{2t} + 1 - \frac{5}{2}$

$$\Rightarrow \quad \boxed{y(t) = \frac{5}{2} e^{2t} - \frac{3}{2}}$$

Example: Use the Picard iteration to solve

$$y' = 2t^4 y, \qquad y(0) = 1.$$

Sol:

$$\int_0^t y'(s)\,ds = \int_0^t 2s^4 y(s)\,ds$$

$$y(t) - y(0) = \qquad\qquad y(0) = 1 \Rightarrow$$

$$\Rightarrow \boxed{y(t) = 1 + 2\int_0^t s^4 y(s)\,ds}$$

$$\boxed{y_{n+1}(t) = 1 + 2\int_0^t s^4 y_n(s)\,ds} \qquad n \geq 0.$$

$$\boxed{y_0 = y(0)}$$

$$y_1(t) = 1 + 2\int_0^t s^4 (1)\,ds \Rightarrow \boxed{y_1 = 1 + \frac{2}{5}t^5}$$

$$y_2(t) = 1 + 2\int_0^t s^4 \left(1 + \frac{2}{5}s^5\right) ds$$

$$= 1 + 2\int_0^t \left(s^4 + \frac{2}{5}s^9\right) ds$$

$$\boxed{y_2 = 1 + \frac{2}{5}t^5 + \frac{2^2}{5}\frac{t^{10}}{10}}$$

$$y_3 = 1 + 2\int_0^t s^4 \left(1 + \frac{2}{5}s^5 + \frac{2^2}{5^2}\frac{1}{2}s^{10}\right) ds$$

$$= 1 + 2\int_0^t \left(s^4 + \frac{2}{5}s^9 + \frac{2}{5^2}s^{14}\right) ds$$

$$\boxed{y_3 = 1 + \frac{2}{5}t^5 + \frac{2^2}{5}\frac{t^{10}}{10} + \frac{2^2}{5^2}\frac{t^{15}}{15}}$$

Recall: $\boxed{e^x - 1 = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k!}}$. Find x

Rewrite : $y_3 = 1 + \underbrace{\frac{2}{5} t^5 + \frac{2^2}{5} \frac{t^{10}}{10} + \frac{2^2}{5^2} \frac{t^{15}}{15}}$

Start reordering factors to get $k!$

$$y_3 = 1 + \frac{2}{5} \frac{t^5}{1!} + \frac{2^2}{5^2} \frac{t^{10}}{2!} + \frac{2^2}{5^2} \frac{1}{5} \frac{2}{2} \frac{t^{15}}{3}$$

$$y_3 = 1 + \left(\frac{2}{5}\right) \frac{t^5}{1!} + \left(\frac{2}{5}\right)^2 \frac{t^{10}}{2!} + \left(\frac{2}{5}\right)^3 \frac{t^{15}}{3!} \quad \checkmark$$

Now rewrite the power of t so they match the index in the factorial

$$y_3 = 1 + \left(\frac{2}{5}\right)^1 \frac{(t^5)^1}{1!} + \left(\frac{2}{5}\right)^2 \frac{(t^5)^2}{2!} + \left(\frac{2}{5}\right)^3 \frac{(t^5)^3}{3!}$$

$$y_3 = 1 + \left(\frac{2}{5} t^5\right)^1 \frac{1}{1!} + \left(\frac{2}{5} t^5\right)^2 \frac{1}{2!} + \left(\frac{2}{5} t^5\right)^3 \frac{1}{3!}$$

Therefore, we guess the limit $k \to \infty$

$$y_\infty = 1 + \left(\frac{2}{5} t^5\right)^1 \frac{1}{1!} + \left(\frac{2}{5} t^5\right)^2 \frac{1}{2!} + \dots + \left(\frac{2}{5} t^5\right)^k \frac{1}{k!} + \dots$$

$$\left.\begin{array}{l} y_\infty = \displaystyle\sum_{k=0}^{\infty} \left(\frac{2}{5} t^5\right)^k \frac{1}{k!} \ ; \\ \text{Recall : } \ e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} \ ; \quad \text{choose } x = \frac{2}{5} t^5 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{y_\infty(t) = e^{\frac{2}{5} t^5}}$$