

The Matrix Exponential

We prove useful properties of the exponential function of a matrix

Objectives

To learn a few properties the exponential of a matrix.

Introduction

Recall that the exponential of a matrix can be defined as an infinite sum,

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots .$$

In the case of a diagonal matrix, we have the formula

$$e^{\text{diag}[d_1, \dots, d_n]} = \text{diag}[e^{d_1}, \dots, e^{d_n}].$$

In the case that A is diagonalizable, with $A = PDP^{-1}$, with D diagonal, the exponential is given by

$$e^{PDP^{-1}} = Pe^DP^{-1}.$$

Also recall the trace of a square matrix is the sum of its diagonal elements,

$$\text{tr} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11} + \cdots + a_{nn}.$$

Properties of the Exponential Matrix

Question 1: (*2 points*)

(a) If $A^2 = A$, then show that

$$e^A = I + (e - 1)A.$$

(b) If $A^2 = I$, then show that

$$2e^A = \left(e + \frac{1}{e}\right)I + \left(e - \frac{1}{e}\right)A.$$

Question 2: (*2 points*)

1. Compute e^A for $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, where a, b are any real constants.
2. By direct computation show that $e^{(A+B)} \neq e^A e^B$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Question 3: (*2 points*) Show that, if A is diagonalizable, then

$$\det(e^A) = e^{\text{tr}(A)}.$$

Notes:

- Use the determinant properties, such as $\det(AB) = \det(A) \det(B)$.
- The formula in this problem is true for all square matrices, but it is hard to prove for nondiagonalizable matrices.

Question 4: (*2 points*) If λ and \mathbf{v} are an eigenvalue and eigenvector of A , then show that

$$e^A \mathbf{v} = e^\lambda \mathbf{v}.$$

Question 5: (*2 points*) Prove the following: If A, B are $n \times n$ matrices such that $AB = BA$, then

$$A e^B = e^B A \quad \text{and} \quad e^A e^B = e^B e^A.$$