Functions Defined by Differential Equations

A differential equation can be used to define new functions

Objectives

To understand how a differential equation can be used to determine properties of its solutions.

Introduction

More often than not, solutions to differential equations cannot be written in terms of previously known functions. When that happens the we say that such solutions define a new type of functions.

Whenever one gets a new function, it is important to characterize such function. This is specially important when the function is not defined by its values, as it happens when the function is defined as a solution of a differential equation. To characterize a function means to find as many properties as possible of such function. When the function is defined as a solution of a differential equation, we need to use that equation to characterize the solution.

In this project we want to learn how one can get properties of a new function using the differential equation this function is solution of. But instead of working with a new function, we will work with well known functions. And we will see that several properties we know about these functions can be obtained from a differential equation that has these functions as its solutions.

The Characterization of functions C and S

We are now going to find the properties of two functions, C and S, solutions of two initial values problems.

Definition 1. Let the function C be the unique solution of the initial value problem

$$C'' + C = 0, \qquad C(0) = 1, \quad C'(0) = 0,$$

and let the function S be the unique solution of the initial value problem

$$S'' + S = 0, \qquad S(0) = 0, \quad S'(0) = 1.$$

Question 1: *(2 points)* Show that the functions C and S are linearly independent—not proportional to each other.

Hint: Recall the properties of the Wronskian of two functions.

Question 2: *(2 points)* Show that the function C is even and the function S is odd.

Note: Recall that a function F is even iff $f(-x) = f(x)$, while a function g is odd iff $g(-x) = -g(x)$.

Hint: Find what initial value problem satisfy the functions $\hat{C}(x) = C(-x)$ and $\hat{S}(x) = S(-x)$. And recall the uniqueness results for initial value problems.

Question 3: *(2 points)* Prove the following relations between the functions C and S,

$$C'(x) = -S(x), \qquad S'(x) = C(x).$$

Hint: Again, recall the uniqueness results for initial value problems.

Question 4: *(2 points)* Show that the functions C and S satisfy the Pythagoras' theorem,

$$C^2(x) + S^2(x) = 1 \qquad \text{for all} \quad x.$$

Hint: Recall Abel's Theorem, which is about the Wronskian of two solutions to a second order differential equation, and use the result of question 3.

Question 5: *(2 points)* Show that the power series expansion of the functions C centered at $x = 0$ is

$$C(x) = \sum_{k=0}^{\infty} (-1)^k \, \frac{x^{2k}}{(2k)!}.$$

Note: A similar calculation can be done for the function S, the result is

$$S(x) = \sum_{k=0}^{\infty} (-1)^k \, \frac{x^{2k+1}}{(2k+1)!},$$

but you do not need to compute it.

Question 6: *(0 points)* What is the well known name for the functions C and S?