Reduction of Order Method

Solving a second order equation by solving two first order ones

Objectives

To understand how to obtain and how the idea of the Reduction of Order Method to solve second order, linear, homogeneous, differential equations.

Introduction

The reduction of order method provides a way to find a solution of a second order, linear, homogeneous differential equation if we already know one solution to that equation. The known solution plays a crucial role to transform the original problem into a new simple problem. The original problem is to solve a second order differential equation. The new problem is to solve two first order equations, one after the other. This is the origin of the name "reduction of order".

This method can be generalized to higer order differential equation as long as they are homogeneous.

The Reduction of Order Method

Theorem 1 (Reduction of Order)**.** If a nonzero function y_1 is solution to

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0.$$

where a_1, a_0 are given functions, then a second solutiony_2 can be written as

$$y_2(t) = v(t)\, y_1(t),$$

where the function v is solution of the equation

$$v'' + \Big(2\frac{y_1'(t)}{y_1(t)} + a_1(t)\Big)\, v' = 0.$$

Furthermore, the equation above for v can be solved and the function y_2 given by

$$y_2(t) = y_1(t) \int \frac{e^{-A_1(t)}}{y_1^2(t)}\, dt,$$

with $A_1(t) = \int a_1(t)\, dt$, is a solution to the original differential equation, not proportional to y_1.

Remark: The differential equation fo v contains v'' and v' but it does not contain v explicitly. Therefore, this is actually a first order equation for for $w = v'$. This is the reason for the name of the method, "reduction of order". The equation for w is linear and first order, so we can solve it using the integrating factor method. One more integration gives v, and then we get y_2.

The Proof of Theorem 1

Question 1: *(3 points)* Prove the first part of Theorem 1: If one writes $y_2(t) = v(t)\, y_1(t)$, then the function v satisfies the equation

$$v'' + \left(2\frac{y_1'(t)}{y_1(t)} + a_1(t)\right) v' = 0.$$

□

Question 2: *(3 points)* Prove the second part of Theorem 1: The differential equation for function v can be solved and the function $y_2(t) = v(t)\, y_1(t)$ given by

$$y_2(t) = y_1(t) \int \frac{e^{-A_1(t)}}{y_1^2(t)}\, dt,$$

with $A_1(t) = \int a_1(t)\, dt$, is a solution to the original differential equation, not proportional to y_1.

Question 3: *(4 points)* Find a second solution y_2 linearly independent to the solution $y_1(t) = t$ of the differential equation

$$t^2 y'' + 3ty' - 3y = 0, \qquad t > 0.$$