Variation of Parameters

A formula for the particular solution of nonhomogeneous equations

Objectives

To understand how to obtain and how to use a formula to find solutions of second order, linear, nonhomogenous differential equations.

Introduction

The general solution theorem nonhomogeneous equations says that the general solution of $L(y) = f$, where $L(y) = y'' + a_1\, y' + a_0\, y$, is

$$y = c_1\, y_1 + c_2\, y_2 + y_p,$$

where

$$L(y_1) = 0, \qquad L(y_2) = 0, \quad \text{and} \quad L(y_p) = f.$$

So, to find all solutions of the nonhomogeneous equation above we need three functions: two fundamental solutions of the homogeneous equation, y_1, y_2, and one solution of the nonhomogeneous equation y_p. The Variation of Parameters Method provides a formula for y_p in terms of y_1, y_2 and f.

Unlike the Undetermined Coefficients Method (UCM), which is a way to guess y_p, the Variation of Parameters Method (VPM) gives a formula to y_p. The Variation of Parameters Method applies to more variable coefficient equations, which are more general than the constant coefficient equations where we can guess y_p.

The Variation of Parameters Method

The variation of parameters formula can be summarized in the following theorem.

Theorem 1 (Variation of Parameters)**.** A particular solution to the equation

$$L(y) = f,$$

with $L(y) = y'' + a_1(t)\, y' + a_0(t)\, y$ and a_1, a_0, f continuous functions, is given by

$$y_p = u_1 y_1 + u_2 y_2,$$

where y_1, y_2 are fundamental solutions of $L(y) = 0$ and u_1, u_2 are

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{12}(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{12}(t)}\, dt,$$

where W_{12} is the Wronskian of y_1 and y_2.

Remarks:

- Recall that the **Wronskian** of functions y_1 and y_2 is

$$W_{12}(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- If y_1 and y_2 are fundamental solutions of $y'' + a_1(t)\, y' + a_0(t)\, y = 0$, then $W_{12}(t) \neq 0$ for all t.

Question 1: *(4 points)* Prove Theorem 1:

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Question 2: *(2 points)* Show that the integration constants in u_1, u_2 can always be chosen zero.

Hint: Choose the integrations constants nonzero and see how they affect the particular solution y_p.

Question 3: *(4 points)* Find the general solution of the nonhomogeneous equation

$$y'' + 4y = -5\csc(2t).$$