Abel and the Wronskian

We study the Wronskian properties including Abel's Theorem

Objectives

Students should know what is a Wronskian of two functions, and what equation this Wronskian satisfies in the case that the two functions are fundamental solutions of a second order linear homogeneous differential equation.

The Wronskian of Two Functions

The **Wronskian** of two differentiable functions y_1, y_2 is the function

$$W_{12}(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) \qquad \left(\quad \Rightarrow \quad W_{12} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \right)$$

We start with the following property of the Wronskian.

Theorem 1. If y_1, y_2 are linearly dependent on $I \subset \mathbb{R}$, then $W_{12} = 0$ on I.

Question 1. *(2 points)* Prove Theorem 1.

Question 2. *(2 points)* Give an example to show that the following: If $W_{12}(t) = 0$ for all t, that **does not** imply that y_1, y_2 are linearly dependent.

Question 3. *(2 points)* Use the example above to find the mistake in the following calculation:

$$W_{12} = 0 \quad \Rightarrow \quad y_1 y_2' - y_1' y_2 = 0 \quad \Rightarrow \quad \frac{y_1 y_2' - y_1' y_2}{y_1^2} = 0 \quad \Rightarrow \quad \left(\frac{y_2}{y_1}\right)' = 0 \quad \Rightarrow \quad \left(\frac{y_2}{y_1}\right) = c \quad \Rightarrow \quad y_2(t) = c\, y_1(t)$$

for all t, where c is any fixed constant for all t.

Abel's Theorem

Theorem 2 (Abel)**.** If y_1, y_2 are twice continuously differentiable solutions of

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0, \tag{1}$$

where a_1, a_0 are continuous on $I \subset \mathbb{R}$, then the Wronskian W_{12} satisfies

$$W_{12}' + a_1(t)\, W_{12} = 0.$$

Therefore, for any $t_0 \in I$, the Wronskian W_{12} is given by the expression

$$W_{12}(t) = W_{12}(t_0)\, e^{-A_1(t)},$$

where $A_1(t) = \displaystyle\int_{t_0}^{t} a_1(s)\, ds$.

Question 4. *(2 points)* Prove Abel's Theorem.

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Question 5. *(2 points)* Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$