

The Matrix Exponential

We prove several properties of the exponential function of a matrix

Objectives

To learn a few properties the exponential of a matrix.

Introduction

Recall that the exponential of a matrix can be defined as an infinite sum,

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots .$$

In the case of a diagonal matrix, we have the formula

$$e^{\text{diag}[d_1, \dots, d_n]} = \text{diag}[e^{d_1}, \dots, e^{d_n}].$$

In the case that A is diagonalizable, with $A = PDP^{-1}$, with D diagonal, the exponential is given by

$$e^{PDP^{-1}} = Pe^D P^{-1}.$$

Also recall the trace of a square matrix is the sum of its diagonal elements,

$$\text{tr} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11} + \cdots + a_{nn}.$$

Requirements

Students need to review in the Lecture Notes the section 5.6, “The Matrix Exponential”.

The Exponential of Particular Matrices

Question 1: *(1 point)* Use the definition of the exponential of a matrix to show the following:

$$\text{If } A^2 = A, \text{ then } e^A = I + (e - 1)A.$$

Question 2: (*2 points*) By direct computation show that $e^{(A+B)} \neq e^A e^B$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hint: You may need **Question (1)** to compute $e^{(A+B)}$.

General Properties of the Exponential Matrix

Question 3: (*2 points*) Prove the following: If A is an $n \times n$, diagonalizable matrix, then

$$\det(e^A) = e^{\operatorname{tr}(A)}.$$

Hint: The determinant can be defined for $n \times n$ matrices having the same properties as the determinant of 2×2 matrices studied in the Deep Dive 09, Matrix Algebra.

Question 4: (*1 point*) Prove the following: If λ and \mathbf{v} are an eigenpair of A , then

$$e^A \mathbf{v} = e^\lambda \mathbf{v}.$$

Question 5: (*2 points*) Prove the following: If A, B are $n \times n$ matrices,

$$AB = BA \quad \Rightarrow \quad e^A e^B = e^B e^A.$$

Hints:

- First, prove that $AB = BA$ implies $AB^n = B^n A$.
- Second, prove that $AB = BA$ implies $Ae^B = e^B A$.

Question 6: (1 point) In the Lecture Notes we have proven the following result: If A is an $n \times n$ matrix and s, t are real constants, then

$$e^{As} e^{At} = e^{A(s+t)}.$$

Use this result to prove the following: If A is an $n \times n$ matrix, then

$$(e^A)^{-1} = e^{-A}.$$

Question 7: (1 point) Prove the following: If A is an $n \times n$ matrix, and $t \in \mathbb{R}$, then

$$\frac{d}{dt} e^{At} = A e^{At}.$$