

Power Series Solutions

We use power series to solve differential equations

Objectives

We use power series expansions to find solutions to differential equations. We learn this method with first and second order linear differential equations having constant coefficients. We end this dive using this method to solve the Legendre equation, which is a second order linear equation having **variable** coefficients.

Introduction

We solved second order, linear, homogeneous, **constant coefficients** equations

$$y'' + a_1 y' + a_0 y = 0,$$

by guessing that the solutions have the form $y(x) = e^{rx}$ and finding the appropriate values for the exponent r . Notice that we are using x for the independent variable instead of t , in this way we agree with most of the literature on this subject. Once we have two fundamental solutions, we then have all the solutions to the homogenous equation. However, this guessing method is not useful with **variable coefficient** equations,

$$y'' + a_1(x) y' + a_0(x) y = 0,$$

because the fundamental solutions are too difficult to guess. But, since the solution $y(x)$ is smooth, it has a Taylor series expansion around any point x_0 where $y(x_0)$ is defined,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

The idea of the power series method is to put the expression above into the differential equation, and then find the values of the coefficients a_n . The Power Series method can be summarized as follows:

- (1) Choose an x_0 and write the solution y as a power series expansion centered at a point x_0 ,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

- (2) Introduce the power series expansion above into the differential equation and find a *recurrence relation*—an equation where the coefficient a_n is related to a_{n-1} (and possibly a_{n-2}).
- (3) Solve the recurrence relation—find a_n in terms of a_0 (and possibly a_1).
- (4) If possible, add up the resulting power series for the solution y .

Requirements

Students may need to read Section 3.1, “Solutions Near Regular Points”, on these **Lecture Notes**.

First Order Equations with Constant Coefficients

The problem below involves a first order, constant coefficient, equation. We use this simple equation to practice the Power Series Method. But recall, this method was created to solve variable coefficient equations.

Question 1: Use a power series around the point $x_0 = 0$ to find all solutions y of the equation

$$y' + cy = 0, \quad c \in \mathbb{R}.$$

(1a) (1 point) Find the recurrence relation relating the coefficient a_n with a_{n-1} .

(1b) (1 point) Solve the recurrence relation, that is, find a_n in terms of a_0 .

(1c) (1 point) Write the solution y as a power series one multiplied by a_0 . Then add the power series expression.

Note: Using the integrating factor method we know that the solution is $y(x) = a_0 e^{-cx}$, with $a_0 \in \mathbb{R}$. We want to recover this solution using the Power Series Method.

Second Order Equations with Constant Coefficients

Once again, the problem below involves a second order, constant coefficient, equation. We use this simple equation to practice the Power Series Method. But recall, this method was created to solve variable coefficient equations.

Question 2: Use a power series around the point $x_0 = 0$ to find all solutions y of the equation

$$y'' + y = 0.$$

- (2a) (1 point) Find the recurrence relation relating the coefficient a_n with a_{n-2} .
- (2b) (1 point) Solve the recurrence relation, which in this case means to find a_n in terms of a_0 for n even, and a_n in terms of a_1 for n odd.
- (2c) (1 point) Write the solution y as a combination of two power series, one multiplied by a_1 , the other multiplied by a_0 . Then add both power series expressions.

Note: Guessing the fundamental solutions we know that the solution is $y(x) = a_0 \cos(x) + a_1 \sin(x)$, with $a_0, a_1 \in \mathbb{R}$. We want to recover these solutions using the Power Series Method.

Second Order Equations with Variable Coefficients

In the previous two cases we used power series to solve a differential equation we already knew how to solve with other, simpler, methods. Now, however, we use the power series method to solve a differential equation that cannot be solved with the methods we studied in the previous chapters. The equation is called the Legendre equation. It appears as part of a more complicated set of equations that need to be solved when we try to find the static electric field in situations with spherical symmetry.

Question 3: Find all solutions of the Legendre equation

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0,$$

where l is any real constant, using power series centered at $x_0 = 0$.

(3a) (1 point) Find the recurrence relation relating the coefficient a_n with a_{n-2} .

(3b) (1 point) Solve the recurrence relation to find a_4 , a_2 in terms of a_0 and to find a_5 , a_3 in terms of a_1 .

(3c) (1 point) If we write the solution y as

$$y(x) = a_0 y_0(x) + a_1 y_1(x),$$

then find the first three terms of the series expansions of y_0 and y_1 . These functions y_0 and y_1 are called Legendre functions.

(3d) (1 point) The powers series that define the Legendre functions have infinitely many terms when the constant l is not an integer. But when l is an integer, either y_0 or y_1 have a finite number of nonzero terms—they terminate—and they are just polynomials. The collection of all polynomials—appropriately normalized—are called the Legendre polynomials. Find the first four Legendre polynomials, P_i , for $i = 0, 1, 2, 3$, which are defined as follows,

$$\begin{aligned} P_0(x) &= y_0(x) && \text{for } l = 0 \\ P_1(x) &= y_1(x) && \text{for } l = 1 \\ P_2(x) &= -\frac{1}{2}y_0(x) && \text{for } l = 2 \\ P_3(x) &= -\frac{3}{2}y_1(x) && \text{for } l = 3. \end{aligned}$$

