Exact Equations

We study how to solve Exact and Semi-Exact Differential Equations

Objectives

Students should learn what are exact differential equations. They should also learn how to find solutions to these equations, by rewriting the equation as a total derivative of a potential function.

Requirements

Students need to read section 1.4 "Exact Equations" in these **Lecture Notes**.

Exact Equations

Definition 1. *An* ***exact*** *differential equation for y is*

$$N(t,y)\,y' + M(t,y) = 0$$

where the functions N and M satisfy

$$\partial_t N(t,y) = \partial_y M(t,y)$$

Remark: We use the notation for partial derivatives $\partial_t N = \dfrac{\partial N}{\partial t}$ and $\partial_y M = \dfrac{\partial M}{\partial y}$.

Theorem 2 (Exact Equations)**.** *If the differential equation*

$$N(t,y)\,y' + M(t,y) = 0 \tag{1}$$

is exact, then it can be written as

$$\frac{d\psi}{dt}(t,y(t)) = 0,$$

where ψ is called a ***potential function*** *and satisfies*

$$N = \partial_y \psi, \qquad M = \partial_t \psi.$$

Therefore, the solutions of the exact equation are given in implicit form as

$$\psi(t,y(t)) = c, \qquad c \in \mathbb{R}.$$

Question 1.*(2 points)* Prove Theorem 2.

Hint: Recall from multivariable calculus: If continuously differentiable functions N, M, on t, y, satisfy

$$\partial_t N(t, y) = \partial_y M(t, y),$$

then there exists a twice continuously differentiable function ψ, on t, y, so that

$$\partial_y \psi(t, y) = N(t, y), \qquad \partial_t \psi(t, y) = M(t, y).$$

Question 2. *(2 points)* Find the explicit form of the solution to the initial value problem

$$2t^3 y\, y' + 3t^2 y^2 + 2t = 0, \qquad y(1) = 1, \qquad 0 < t < \sqrt{2}.$$

Question 3. *(2 points)* Find the implicit form of the solution to the initial value problem

$$-3\sin(3y)\,t\,y' + \cos(3y) + 2\cos(2y)\,t^2\,y' + 2\sin(2y)\,t + 9t^2 = 0, \qquad y(1) = 0.$$

Semi-Exact Equations

More often than not a differential equation $N(t, y)\, y' + M(t, y) = 0$ is not exact, that is, $\partial_t N \neq \partial_y M$.

Definition 3. *A **semi-exact** differential equation is an equation of the form $N(t, y)\, y' + M(t, y) = 0$ which is not exact but there exists a function $\mu(t)$, called an **integrating factor**, so that*

$$\mu(t) N(t, y)\, y' + \mu(t) M(t, y) = 0$$

is exact.

Therefore, a semi-exact differential equation can be transformed into an exact equation after a multiplication by an integrating factor, called μ, which is function of t only.

Theorem 4. *If the equation $N(t, y)\, y' + M(t, y) = 0$ is not exact, with $N \neq 0$, and the function*

$$h = \frac{\partial_y M - \partial_t N}{N}$$

depends only on t and not on y, then the equation

$$\mu(t) N(t, y)\, y' + \mu(t) M(t, y) = 0 \tag{2}$$

is exact for a function μ, depending only on t, solution of

$$\mu'(t) = h(t)\, \mu(t).$$

Question 4. *(2 points)* Prove Theorem 4.

Question 5. *(2 points)* Find all the solutions, in implicit form, of the semi-exact differential equation

$$6t\,y^2\,y' + 6y^3 + 5t^2 = 0, \qquad t > 0.$$