The Matrix Exponential

We prove useful properties of the exponential function of a matrix

Objectives

To learn a few properties the exponential of a matrix.

Introduction

Recall that the exponential of a matrix can be defined as an infinite sum,

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots .$$

In the case of a diagonal matrix, we have the formula

$$e^{\operatorname{diag}[d_1,\cdots,d_n]} = \operatorname{diag}\big[e^{d_1},\cdots,e^{d_n}\big].$$

In the case that A is diagonalizable, with $A = PDP^{-1}$, with D diagonal, the exponential is given by

$$e^{PDP^{-1}} = Pe^D P^{-1}.$$

Also recall the trace of a square matrix is the sum of its diagonal elements,

$$\operatorname{tr}\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11} + \cdots + a_{nn}.$$

Requirements

Students need to review in the Lecture Notes the section 5.6, “The Matrix Exponential”.

Properties of the Exponential Matrix

Question 1: *(2 points)*

(a) If $A^2 = A$, then show that

$$e^A = I + (e - 1)\, A.$$

(b) If $A^2 = I$, then show that

$$2\, e^A = \left(e + \frac{1}{e}\right) I + \left(e - \frac{1}{e}\right) A.$$

Question 2: *(2 points)*

(a) Compute e^A for $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, where a, b are any real constants.

(b) By direct computation show that $e^{(A+B)} \neq e^A \, e^B$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Question 3: *(2 points)* Show that, if A is diagonalizable, then

$$\det\big(e^{A}\big) = e^{\operatorname{tr}(A)}.$$

Notes:

- Use the determinant properties, such as $\det(AB) = \det(A)\,\det(B)$.
- The formula in this problem is true for all square matrices, but it is hard to prove for nondiagonalizable matrices.

Question 4: *(2 points)* If λ and $\boldsymbol{v}$ are an eigenvalue and eigenvector of A, then show that

$$e^{A}\boldsymbol{v} = e^{\lambda}\,\boldsymbol{v}.$$

Question 5: *(2 points)* Prove the following: If A, B are $n \times n$ matrices such that $AB = BA$, then

$$A\, e^{B} = e^{B}\, A \quad \text{and} \quad e^{A}\, e^{B} = e^{B}\, e^{A}.$$