Functions Defined by Differential Equations

A differential equation can be used to define new functions

Objectives

To understand how a differential equation can be used to determine properties of its solutions.

Introduction

More often than not, solutions of differential equations cannot be written in terms of previously known functions. When that happens the we say that such solutions define a new type of functions. It is then important to characterize such functions, that is, to find as many properties as possible of such functions. When a function is defined as a solution of a differential equation—instead of defined by the function values—we use that equation to characterize the solution.

In this project we want to learn how to use a differential equation to characterize its solutions. But instead of studying a differential equation having new functions as solutions, we will study a differential equation with well known functions as solutions. Students must find out, by the end of these notes, what functions we have been working with.

Requirements

Students need to review in the Lecture Notes the section 2.1, "General Properties". In particular students should focus on the Existence and uniqueness of solutions theorem and the Wronskian definition and properties.

The Characterization of functions γ and σ

We are now going to find the properties of two functions, γ and σ, solutions of two initial values problems.

Definition 1. Let the function γ be the unique solution of the initial value problem

$$\gamma'' + \gamma = 0, \qquad \gamma(0) = 1, \quad \gamma'(0) = 0,$$

and let the function σ be the unique solution of the initial value problem

$$\sigma'' + \sigma = 0, \qquad \sigma(0) = 0, \quad \sigma'(0) = 1.$$

Question 1: *(2 points)* Show that the functions γ and σ are linearly independent—not proportional to each other.

Hint: Recall the properties of the Wronskian of two functions.

Question 2: *(2 points)* Show that the function γ is even and the function σ is odd.

Note: Recall that a function f is even iff $f(-x) = f(x)$, while a function g is odd iff $g(-x) = -g(x)$.

Hint: Find what initial value problem satisfy the functions $\hat{\gamma}(x) = \gamma(-x)$ and $\hat{\sigma}(x) = \sigma(-x)$. And recall the uniqueness results for initial value problems.

Question 3: *(2 points)* Prove the following relations between the functions γ and σ,

$$\gamma'(x) = -\sigma(x), \qquad \sigma'(x) = \gamma(x).$$

Hint: Again, recall the uniqueness results for initial value problems.

Question 4: *(2 points)* Show that the functions γ and σ satisfy the Pythagoras' theorem,

$$\gamma^2(x) + \sigma^2(x) = 1 \qquad \text{for all} \quad x.$$

Hint: Recall Abel's Theorem, which is about the Wronskian of two solutions to a second order differential equation, and use the result of question 3.

Question 5: *(2 points)* Show that the power series expansion of the functions γ centered at $x = 0$ is

$$\gamma(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

Note: A similar calculation can be done for the function σ, the result is

$$\sigma(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!},$$

but you do not need to compute it.

Question 6: *(0 points)* What is the well known name for the functions γ and σ?