Abel and the Wronskian

We study the Wronskian properties including Abel's Theorem

Objectives

Students should know what is a Wronskian of two functions, and what equation this Wronskian satisfies in the case that the two functions are fundamental solutions of a second order linear homogeneous differential equation.

Requirements

Students will need to read in the Lecture Notes the subsection 2.1.4, "The Wronskian Function", and subsection 2.1.5 "Abel's Theorem".

The Wronskian of Two Functions

The **Wronskian** of two differentiable functions y_1, y_2 is the function

$$W_{12}(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) \qquad \left(\Rightarrow \quad W_{12} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \right)$$

We start with the following property of the Wronskian.

Theorem 1. If y_1, y_2 are linearly dependent on $I \subset \mathbb{R}$, then $W_{12} = 0$ on I.

Question 1.*(2 points)* Prove Theorem 1.

Question 2. *(2 points)* Give an example to show that the following: If $W_{12}(t) = 0$ for all t, that **does not** imply that y_1, y_2 are linearly dependent.

Question 3. *(2 points)* Use the example above to find the mistake in the following calculation:

$$W_{12} = 0 \quad \Rightarrow \quad y_1 y_2' - y_1' y_2 = 0 \quad \Rightarrow \quad \frac{y_1 y_2' - y_1' y_2}{(y_1)^2} = 0 \quad \Rightarrow \quad \left(\frac{y_2}{y_1}\right)' = 0 \quad \Rightarrow \quad \frac{y_2}{y_1} = c \quad \Rightarrow \quad y_2(t) = c\, y_1(t)$$

for all t, where c is any fixed constant for all t.

Abel's Theorem

Theorem 2 (Abel)**.** If y_1, y_2 are twice continuously differentiable solutions of

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0, \tag{1}$$

where a_1, a_0 are continuous on $I \subset \mathbb{R}$, then the Wronskian W_{12} satisfies

$$W_{12}' + a_1(t)\, W_{12} = 0.$$

Therefore, for any $t_0 \in I$, the Wronskian W_{12} is given by the expression

$$W_{12}(t) = W_{12}(t_0)\, e^{-A_1(t)},$$

where $A_1(t) = \displaystyle\int_{t_0}^{t} a_1(s)\, ds$.

Question 4. *(2 points)* Prove Abel's Theorem.
(Try to make your own proof, do not copy line by line from the notes.)

□

Question 5. *(2 points)* Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$