The Matrix Exponential

We prove several properties of the exponential function of a matrix

Objectives

To learn a few properties the exponential of a matrix.

Introduction

Recall that the exponential of a matrix can be defined as an infinite sum,

\[ e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots. \]

In the case of a diagonal matrix, we have the formula

\[ e^{\text{diag}[d_1, \ldots, d_n]} = \text{diag}[e^{d_1}, \ldots, e^{d_n}]. \]

In the case that \( A \) is diagonalizable, with \( A = PDP^{-1} \), with \( D \) diagonal, the exponential is given by

\[ e^{PDP^{-1}} = Pe^{D}P^{-1}. \]

Also recall the trace of a square matrix is the sum of its diagonal elements,

\[ \text{tr} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11} + \cdots + a_{nn}. \]

Requirements

Students need to review in the Lecture Notes the section 5.6, “The Matrix Exponential”. 

The Exponential of Particular Matrices

Question 1:

(a) (1 point) If $A^2 = A$, then show that
\[ e^A = I + (e - 1)A. \]

(b) (1 point) If $A^2 = I$, then show that
\[ 2e^A = \left( e + \frac{1}{e} \right)I + \left( e - \frac{1}{e} \right)A. \]
Question 2: (1 point) By direct computation show that \( e^{(A+B)} \neq e^A e^B \) for

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

**Hint:** You may need **Question (1a)** to compute \( e^{(A+B)} \).
General Properties of the Exponential Matrix

**Question 3:** (1 point) Prove the following: If \( A \) is an \( n \times n \), diagonalizable matrix, then
\[
\det(e^A) = e^{\text{tr}(A)}.
\]

**Hint:** The determinant can be defined for \( n \times n \) matrices having the same properties as the determinant of \( 2 \times 2 \) matrices studied in the Deep Dive 09, Matrix Algebra.

**Question 4:** (1 point) Prove the following: If \( \lambda \) and \( v \) are an eigenpair of \( A \), then
\[
e^A v = e^\lambda v.
\]
Question 5: (1 point) Prove the following: If $A$, $B$ are $n \times n$ matrices,

$$AB = BA \implies e^A e^B = e^B e^A.$$  

Hints:

- First, prove that $AB = BA$ implies $AB^n = B^n A$.
- Second, prove that $AB = BA$ implies $A e^B = e^B A$. 
Question 6: (2 points) Prove the following: If $A$ is an $n \times n$ matrix and $s$, $t$ are real constants, then

$$e^{As} e^{At} = e^{A(s+t)}.$$ 

Hints:

- Write $e^{As} = \left( \sum_{j=0}^{\infty} \frac{A^j s^j}{j!} \right)$ and $e^{At} = \left( \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \right)$, then compute their product.

- Switch from indices $j$ and $k$ to indices $n$ and $k$, where $n = j + k$.

- Recall the binomial formula $(s + t)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} s^{n-k} t^k$. 
Question 7: (1 point) Use the result in Question 6 to prove the following: If $A$ is an $n \times n$ matrix, then

$$(e^A)^{-1} = e^{-A}.$$ 

Question 8: (1 point) Prove the following: If $A$ is an $n \times n$ matrix, and $t \in \mathbb{R}$, then

$$\frac{d}{dt} e^{At} = A e^{At}.$$