The Matrix Exponential

We prove several properties of the exponential function of a matrix

Objectives

To learn a few properties the exponential of a matrix.

Introduction

Recall that the exponential of a matrix can be defined as an infinite sum,

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots .$$

In the case of a diagonal matrix, we have the formula

$$e^{\mathrm{diag}[d_1,\cdots,d_n]} = \mathrm{diag}\big[e^{d_1}, \cdots, e^{d_n}\big].$$

In the case that A is diagonalizable, with $A = PDP^{-1}$, with D diagonal, the exponential is given by

$$e^{PDP^{-1}} = Pe^D P^{-1}.$$

Also recall the trace of a square matrix is the sum of its diagonal elements,

$$\mathrm{tr} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11} + \cdots + a_{nn}.$$

Requirements

Students need to review in the Lecture Notes the section 5.6, “The Matrix Exponential”.

The Exponential of Particular Matrices

Question 1:

(a) *(1 point)* If $A^2 = A$, then show that

$$e^A = I + (e - 1)\, A.$$

(b) *(1 point)* If $A^2 = I$, then show that

$$2\, e^A = \left(e + \frac{1}{e}\right) I + \left(e - \frac{1}{e}\right) A.$$

Question 2: *(1 point)* By direct computation show that $e^{(A+B)} \neq e^A \, e^B$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hint: You may need **Question (1a)** to compute $e^{(A+B)}$.

General Properties of the Exponential Matrix

Question 3: *(1 point)* Prove the following: If A is an $n \times n$, diagonalizable matrix, then

$$\det\big(e^{A}\big) = e^{\operatorname{tr}(A)}.$$

Hint: The determinant can be defined for $n \times n$ matrices having the same properties as the determinant of 2×2 matrices studied in the Deep Dive 09, Matrix Algebra.

Question 4: *(1 point)* Prove the following: If λ and $\boldsymbol{v}$ are an eigenpair of A, then

$$e^{A}\boldsymbol{v} = e^{\lambda}\,\boldsymbol{v}.$$

Question 5: *(1 point)* Prove the following: If A, B are $n \times n$ matrices,

$$AB = BA \quad \Rightarrow \quad e^A\, e^B = e^B\, e^A.$$

Hints:

- First, prove that $AB = BA$ implies $A\, B^n = B^n\, A$.
- Second, prove that $AB = BA$ implies $A\, e^B = e^B\, A$.

Question 6: *(2 points)* Prove the following: If A is an $n \times n$ matrix and s, t are real constants, then

$$e^{As}\, e^{At} = e^{A(s+t)}.$$

Hints:

- Write $e^{As} = \left(\sum_{j=0}^{\infty} \frac{A^j\, s^j}{j!}\right)$ and $e^{At} = \left(\sum_{k=0}^{\infty} \frac{A^k\, t^k}{k!}\right)$, then compute their product.
- Switch from indices j and k to indices n and k, where $n = j + k$.
- Recall the binomial formula $(s+t)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} s^{n-k} t^k$.

Question 7: *(1 point)* Use the result in **Question 6** to prove the following: If A is an $n \times n$ matrix, then

$$\left(e^{A}\right)^{-1} = e^{-A}.$$

Question 8: *(1 point)* Prove the following: If A is an $n \times n$ matrix, and $t \in \mathbb{R}$, then

$$\frac{d}{dt} e^{At} = A\, e^{At}.$$