Matrix Algebra

We prove useful properties of determinants and inverse matrices

Objectives

To learn a few properties the inverse of 2×2 matrices and their determinants.

Introduction

Recall that the determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$, and the inverse matrix is

$$(A^{-1}) = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \text{in the case that} \quad \det(A) \neq 0.$$

This matrix (A^{-1}) is called the inverse of matrix A because

$$(A^{-1})\,A = I, \qquad A\,(A^{-1}) = I, \quad \text{where} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Also recall that the trace of matrix is the sum of its diagonal elements, so the trace of A above is

$$\operatorname{tr}(A) = a + d.$$

Requirements

Students need to review in the Lecture Notes the section 5.3, “Matrix Algebra”.

Properties of Determinants

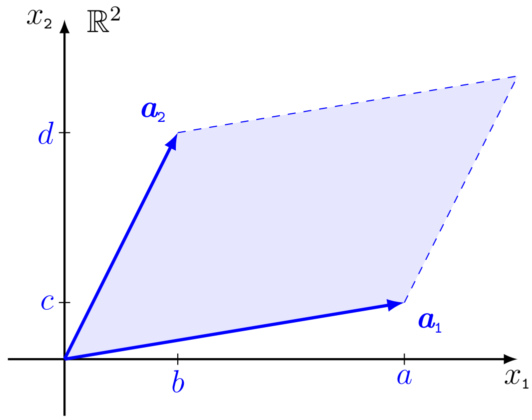
Question 1: *(1 point)* Prove that every invertible 2×2 matrices A, B, satisfy $\det(AB) = \det(A)\det(B)$.

Question 2: *(1 point)* Determine whether the equation $\det(A + B) = \det(A) + \det(B)$ is true or not. If it is true, prove it for all 2×2 matrices A and B; if it is not true, give an example.

Question 3: *(2 points)* Denote a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in terms of its column vectors as $A = \begin{bmatrix} \boldsymbol{a}_1, \boldsymbol{a}_2 \end{bmatrix}$. Suppose that the vectors $\boldsymbol{a}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\boldsymbol{a}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$ are given in the figure below. Use that picture to prove

$$\text{Area of the shaded parallelogram} = |\det(A)|.$$

Hint: Relate the parallelogram area with areas you can easily compute, such as triangle and rectangle areas.



Properties of Inverse Matrices

Question 4: *(1 point)* Prove that for every invertible 2×2 matrix holds that $\left((A^{-1})^{-1}\right) = A$.

Question 5: *(1 point)* Prove that every invertible 2×2 matrix satisfy $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Question 6: *(1 point)* Prove that every invertible 2×2 matrices A, B, satisfy $(AB)^{-1} = (B^{-1})(A^{-1})$.

Question 7: *(2 points)* Prove the following statements.

(7a) If a square matrix A satisfies $A^2 = 0$, then the matrix $(I - A)$ is invertible. Find the inverse of $(I - A)$.

(7b) If a square matrix A satisfies $A^3 = 0$, then the matrix $(I - A)$ is invertible. Find the inverse of $(I - A)$.

Cayley-Hamilton Theorem

Question 7: *(1 point)* Show that every 2×2 matrix A satisfies the following **matrix equation**,

$$A^2 - \operatorname{tr}(A)\, A + \det(A)\, I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$