Power Series Solutions

We use power series to solve second order differential equations

Objectives

We use power series expansions to find solutions to second order, linear, variable coefficient equations

Introduction

We solved second order, linear, homogeneous, **constant coefficients** equations

$$y'' + a_1\, y' + a_0\, y = 0,$$

by guessing that the solutions have the form $y(x) = e^{rx}$—here we use x for the independent variable instead of t—and finding the appropriate values for the exponent r. Once we have two fundamental solutions, we then have all the solutions to the homogenous equation. However, this guessing method is not useful with **variable coefficient** equations,

$$y'' + a_1(x)\, y' + a_0(x)\, y = 0,$$

because the fundamental solutions are too difficult to guess. But, since the solution $y(x)$ is smooth, it has a Taylor series expansion around any point x_0 where $y(x_0)$ is defined,

$$y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^n.$$

The idea of the power series method is to put the expression above into the differential equation, and then find the values of the coefficients a_n. The Power Series method can be summarized as follows:

(1) Choose an x_0 and write the solution y as a power series expansion centered at a point x_0,

$$y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^n.$$

(2) Introduce the power series expansion above into the differential equation and find a *recurrence relation*—an equation where the coefficient a_n is related to a_{n-1} (and possibly a_{n-2}).

(3) Solve the recurrence relation—find a_n in terms of a_0 (and possibly a_1).

(4) If possible, add up the resulting power series for the solution y.

Requirements

Students may need to read Section 3.1, “Solutions Near Regular Points”, on these **Lecture Notes**.

Power Series on First Order Equations

The problem below involves a first order, constant coefficient, equation. We use this simple equation to practice the Power Series Method. But recall, this method was created to solve variable coefficient equations.

Question 1: Use a power series around the point $x_0 = 0$ to find all solutions y of the equation

$$y' + c\,y = 0, \qquad c \in \mathbb{R}.$$

(1) *(2 points)* Find the recurrence relation relating the coefficient a_n with a_{n-1}.

(2) *(2 points)* Solve the recurrence relation, that is, find a_n in terms of a_0.

(3) *(1 points)* Write the solution y as a power series one multiplied by a_0. Then add the power series expression.

Note: Using the integrating factor method we know that the solution is $y(x) = a_0\, e^{-cx}$, with $a_0 \in \mathbb{R}$. We want to recover this solution using the Power Series Method.

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Power Series on Second Order Equations

Once again, the problem below involves a second order, constant coefficient, equation. We use this simple equation to practice the Power Series Method. But recall, this method was created to solve variable coefficient equations.

Question 2: Use a power series around the point $x_0 = 0$ to find all solutions y of the equation

$$y'' + y = 0.$$

(1) *(2 points)* Find the recurrence relation relating the coefficient a_n with a_{n-2}.

(2) *(2 points)* Solve the recurrence relation, which in this case means to find a_n in terms of a_0 for n even, and a_n in terms of a_1 for n odd.

(3) *(1 points)* Write the solution y as a combination of two power series, one multiplied by a_1, the other multiplied by a_0. Then add both power series expressions.

Note: Guessing the fundamental solutions we know that the solution is $y(x) = a_0 \cos(x) + a_1 \sin(x)$, with $a_0, a_1 \in \mathbb{R}$. We want to recover these solutions using the Power Series Method.

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