Variation of Parameters

A formula for the particular solution of non-homogeneous equations

Objectives

To understand how to obtain and how to use a formula to find solutions of second order, linear, nonhomogenous differential equations.

Introduction

The general solution theorem nonhomogeneous equations says that the general solution of $L(y) = f$, where $L(y) = y'' + a_1\, y' + a_0\, y$, is

$$y = c_1\, y_1 + c_2\, y_2 + y_p,$$

where

$$L(y_1) = 0, \qquad L(y_2) = 0, \quad \text{and} \quad L(y_p) = f.$$

So, to find all solutions of the nonhomogeneous equation above we need three functions: two fundamental solutions of the homogeneous equation, y_1, y_2, and one solution of the nonhomogeneous equation y_p. The Variation of Parameters Method provides a formula for y_p in terms of y_1, y_2 and f.

Unlike the Undetermined Coefficients Method (UCM), which is a way to guess y_p, the Variation of Parameters Method (VPM) gives a formula to y_p. The Variation of Parameters Method applies to more variable coefficient equations, which are more general than the constant coefficient equations where we can guess y_p.

Requirements

Students will need to read in the Lecture Notes the subsection 2.3.3, "The Variation of Parameters Method".

The Variation of Parameters Method

The variation of parameters formula can be summarized in the following theorem.

Theorem 1 (Variation of Parameters)**.** A particular solution to the equation

$$L(y) = f,$$

with $L(y) = y'' + a_1(t)\, y' + a_0(t)\, y$ and a_1, a_0, f continuous functions, is given by

$$y_p = u_1 y_1 + u_2 y_2,$$

where y_1, y_2 are fundamental solutions of $L(y) = 0$ and u_1, u_2 are

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{12}(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{12}(t)}\, dt,$$

where W_{12} is the Wronskian of y_1 and y_2.

Remarks:

- Recall that the **Wronskian** of functions y_1 and y_2 is

$$W_{12}(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- If y_1 and y_2 are fundamental solutions of $y'' + a_1(t)\, y' + a_0(t)\, y = 0$, then $W_{12}(t) \neq 0$ for all t.

Question 1: *(2 points)* Prove Theorem 1:

Note: Make your own proof. Line by line copy from the notes won't get you any credit.

Question 2: *(2 points)* Show that the integration constants in u_1, u_2 can always be chosen zero.

Hint: Choose the integrations constants nonzero and see how they affect the particular solution y_p.

Question 3: *(2 points)* Find the general solution of the non-homogeneous equation

$$y'' - 6y' + 9y = \frac{e^{3t}}{t^2}, \qquad t > 0.$$

Question 4: *(2 points)* Find a particular solution to the differential equation

$$t^2y'' - 2y = 3t^2 - 1,$$

knowing that $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2y'' - 2y = 0$.

Question 5: *(2 points)* Find the general solution of the non-homogeneous equation

$$y'' + 4y = -5\csc(2t).$$