The Bernoulli Equation

We transform a nonlinear equation into a linear equation

Objectives

Students should be able to identify and solve a Bernoulli equation.

Introduction

The **Bernoulli equation** with coefficients functions p, q, and index $n \in \mathbb{R}$ is given by

$$y' = p(t)\, y + q(t)\, y^n. \tag{1}$$

Remarks:

- For $n \neq 0, 1$ the equation is nonlinear.
- If $n = 2$ we get the *logistic equation*. Indeed, choosing $p = r$ and $q = -r/K$ we get
$$y' = ry\Big(1 - \frac{y}{K}\Big).$$
- The Bernoulli equation in this sections is not the Bernoulli equation from fluid dynamics.
- The Bernoulli equation is special because *it is a nonlinear equation that can be transformed into a linear equation*.

Requirements

Students will need to read the first part of section 1.5 in both the Lecture Notes Companion and the Lecture Notes. Then they need to read the last part of section 1.5 in the Lecture Notes, "The Bernoulli Equation".

Theorem 1. The function y is a solution of the Bernoulli equation

$$y' = p(t)\, y + q(t)\, y^n, \qquad n \neq 1,$$

iff the function $v = 1/y^{(n-1)}$ is solution of the linear differential equation

$$v' = -(n-1)p(t)\, v - (n-1)q(t).$$

Question 1. *(2 points)* Prove Theorem 1.

Question 2. *(2 points)* Find every nonzero solution of the differential equation

$$y' = y + 3\,y^4.$$

Question 3. *(2 points)* Find every nonzero solution of the constant coefficients Bernoulli equation

$$y' = p\,y + q\,y^n, \qquad n \neq 0, 1,$$

where p, q are constants. Write the implicit form of the solution as

$$\frac{1}{y^{n-1}} = f(t, n, p, q, c)$$

where c is an integration constant. Find the right-hand side above, $f(t, n, p, q, c)$.

Question 4. *(4 points)* Find every solution of the *variable coefficients* equation $t\, y' = 3y + \dfrac{9}{2}\, t^5\, y^{1/3}$.