Companion for MSU's MTH 235 Lecture Notes

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Contents

1.1. Bacteria Reproduce like Rabbits

Section Objective(s):

- Overview of Differential Equations.
- The Difference Equation.
- The Continuum Equation.
- Summary and Consistency.

1.1.1. Overview of Differential Equations.

Remarks:

(a) A differential equation is an equation, the unknown is a function, and both the function and its derivatives may appear in the equation.

(b) Differential equations are essential for a mathematical description of nature.

(c) In this section we show that differential equations can be obtained from difference equations.

(d) We focus on a specific problem—a quantitative description of bacteria growth under certain conditions including unlimited space and food.

1.1.2. The Difference Equation.

The Problem: We want to know how bacteria grow in time when they have

unlimited space and food supplies.

The Experiments:

(1) **First Experiment:** We put an $P(0)$ bacteria in a small region at the center of a petri dish, which is full bacteria food.

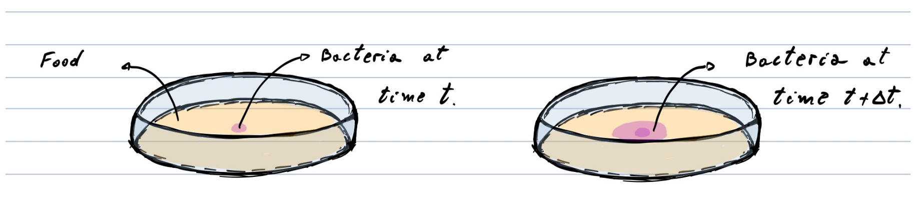

FIGURE 1. Bacteria growth experiment with unlimited food and space.

(2) We measure the bacteria population after regular time intervals.

- The time interval between measurements is $\Delta t_1 = 1$ hour.
- Denote the bacteria population after n time intervals by $P(n\Delta t_1) = P(n)$,
- Introduce the initial bacteria population $P(0) = P(0)$,

(3) Our first n measurements are the following,

$$
\begin{aligned}
P(1) &= P_0 + \Delta P_1, & \Delta P_1 &= K_1\, P_0,\\
P(2) &= P(1) + \Delta P_2, & \Delta P_2 &= K_1\, P(1),\\
&\ \vdots & &\ \vdots\\
P(n) &= P(n-1) + \Delta P_n, & \Delta P_n &= K_1\, P(n-1),
\end{aligned}
$$

Summary so far:

$$P(n\Delta t_1) = P((n-1)\Delta t_1) + K_1\, P((n-1)\Delta t_1), \qquad K_1 \quad \text{depends on the bacteria.}$$

(4) **Second Experiment:** We reduce the time interval to $\Delta t_2 = \dfrac{1}{2} = \dfrac{\Delta t_1}{2}$ when we take measurements. We find:

$$P(n\Delta t_2) = P((n-1)\Delta t_1) + K_2\, P((n-1)\Delta t_2), \qquad n = 1, 2, \cdots, N,$$

where $K_2 = \dfrac{K_1}{2}$.

(5) Experiment m-th: We use a time interval $\Delta t_m = \dfrac{\Delta t_1}{m}$. We get

$$P(n\Delta t_m) = P((n-1)\Delta t_m) + K_m\, P((n-1)\Delta t_m), \qquad n = 1, 2, \cdots, N,$$

where $K_m = \dfrac{K_1}{m}$. Therefore,

$$K_m = \frac{K_1}{m} \quad \Rightarrow \quad K_m = \frac{K_1}{\Delta t_1}\,\frac{\Delta t_1}{m} \quad \Rightarrow \quad K_m = r\,\Delta t_m, \quad \text{where} \quad r = \frac{K_1}{\Delta t_1}.$$

The constant r depends only on the type of bacteria.

(6) Summary: If we drop the subindex m, we get

$$K = r\,\Delta t,$$

where Δt is any time interval. Therefore, the final conclusion of all our experiments is the following: The population of bacteria $P(n\Delta t)$ after $n \geqslant 1$ time intervals $\Delta t > 0$ is given by the difference equation

$$P(n\Delta t) = P((n-1)\Delta t) + r\,\Delta t\, P((n-1)\Delta t),$$

where $r > 0$ is a constant that depends on the particular type of bacteria.

1.1.3. Solving the Difference Equation.

The difference equation relates $\underline{P(n\Delta t)\qquad\qquad}$ with $\underline{P((n-1)\Delta t)\qquad\qquad\qquad}$.

To solve the difference equation means to relate $\underline{P(n\Delta t)\qquad\qquad}$ with $\underline{P(0)\qquad\qquad}$.

The difference equation above can be solved, and the result is summarized below.

Theorem 2. The difference equation

$$P(n\Delta t) = P((n-1)\Delta t) + r\,\Delta t\,P((n-1)\Delta t),$$

relating $\underline{P(n\Delta t)\qquad\qquad}$ with $\underline{P((n-1)\Delta t)\qquad\qquad\qquad}$ has the solution

$$P(n\Delta t) = (1 + r\,\Delta t)^n\,P(0),$$

relating $\underline{P(n\Delta t)\qquad\qquad}$ with $\underline{P(0)\qquad\qquad}$.

Proof: We now that:

$$P(n\Delta t) = (1 + r\,\Delta t)\,P((n-1)\Delta t),$$

but

$$P((n-1)\Delta t) = (1 + r\,\Delta t)\,P((n-2)\Delta t),$$

and so on till we reach P_0. Therefore,

$$\begin{aligned} P(n\Delta t) &= (1 + r\,\Delta t)\,P((n-1)\Delta t) \\ &= (1 + r\,\Delta t)^2\,P((n-2)\Delta t) \\ &\;\vdots \\ &= (1 + r\,\Delta t)^n\,P_0. \end{aligned}$$

So, the solution of the discrete equation is

$$P(n\Delta t) = (1 + r\,\Delta t)^n\,P_0.$$

$\square$

1.1.4. The Continuum Equation.

We study the difference population equation and its solutions in the continuum limit:

$$\Delta t \to 0, \quad \text{such that} \quad n\,\Delta t = t > 0, \quad \text{is constant.}$$

Hence $n = \dfrac{t}{\Delta t} \to \infty$. The result is:

Theorem 3. The continuum limit of the difference equation

$$P(n\Delta t) = P((n-1)\Delta t) + r\,\Delta t\,P((n-1)\Delta t),$$

is the differential equation

$$P'(t) = r\,P(t).$$

Remark: The differential equation is called the exponential growth equation.

Proof: We start renaming n as $n+1$, so the discrete equation is

$$P((n+1)\Delta t) = P(n\Delta t) + r\,\Delta t\,P(n\Delta).$$

From here it is simple to see that

$$P(n\Delta t + \Delta t) - P(n\Delta t) = r\,\Delta t\,P(n\Delta).$$

We use that $n\,\Delta t = t$, then

$$P(t + \Delta t) - P(t) = r\,\Delta t\,P(t).$$

Dividing by Δt we get

$$\frac{P(t+\Delta t) - P(t)}{\Delta t} = r\,P(t).$$

The continuum limit is

$$\lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = r\,P(t).$$

Since t is held constant and $\Delta t \to 0$, the left-hand side above is P',

$$P'(t) = \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} \quad \Rightarrow \quad P'(t) = r\,P(t)$$

$\square$

1.1.5. Solving the Continuum Equation.

Theorem 4. There is only one solution P to the initial value problem

$$P'(t) = r\,P(t), \qquad P(0) = P_0,$$

where P_0 is a constant, given by

$$P(t) = P_0\,e^{rt}.$$

Proof: Divide the differential equation by P,

$$\frac{P'(t)}{P(t)} = r.$$

We now integrate both sides with respect to time,

$$\int \frac{P'(t)}{P(t)}\,dt = \int r\,dt.$$

The integral on the right-hand side is simple to do, we need to integrate a constant,

$$\int \frac{P'(t)}{P(t)}\,dt = rt + c_0,$$

where c_0 is an arbitrary constant. On the left-hand side we can introduce a substitution

$$p = P(t) \quad \Rightarrow \quad dp = P'(t)\,dt.$$

Then, the the equation above becomes

$$\int \frac{dp}{p} = rt + c_0.$$

The integral above is simple to do and the result is

$$\ln|p| = rt + c_0.$$

We now replace back $p = P(t)$, and we can solve for P,

$$\ln|P(t)| = rt + c_0 \quad \Rightarrow \quad |P(t)| = e^{rt+c_0} = e^{kt}\,e^{c_0} \quad \Rightarrow \quad P(t) = (\pm e^{c_0})\,e^{rt}.$$

Denote $c = (\pm e^{c_0})$, then all the solutions to the exponential growth equation are,

$$P(t) = c\, e^{rt}, \qquad c \in \mathbb{R}.$$

We now use the *initial condition*, $P(0) = P_0$,

$$P_0 = P(0) = c\, e^0 = c \quad \Rightarrow \quad c = P_0,$$

So we get $P(t) = P_0\, e^{rt}$. $\square$

1.1.6. Summary and Consistency.

We can summarize all this in the following picture

Difference description	$\Delta t \to 0$	**Continuous description**
$P(n\Delta t) = (1 + r\,\Delta t)\, P((n-1)\Delta t)$	$\longrightarrow$	$P'(t) = r\, P(t)$
$\downarrow$		$\downarrow$
Soving the equation		Solving the equation
$\downarrow$		$\downarrow$
$P(n\Delta t) = (1 + r\,\Delta t)^n\, P_0$	$\xrightarrow{\text{Consistency}}$	$P(t) = P_0\, e^{rt}$

Theorem 5. (Consistency) The continuum limit ______ of the solutions of the difference population equations are the solutions of the continuum population equation,

$$P(n\Delta t) = (1 + r\,\Delta t)^n\, P_0 \quad \longrightarrow \quad P(t) = P_0\, e^{rt}.$$

(The proof is in the Lecture Notes.)

1.2. Introduction to Modeling

Section Objective(s):

Part 1:

- Linear Growth and Decay
- Exponential Growth and Decay.

Part 2:

- Migration Terms.
- The Logistic Equation.
- Interacting Species.

Remarks:

- Modeling is a mathematical description of a physical system using differential equations.
- Linear models are models such that their solutions contain linear functions of the independent variable.
- Exponential models are models such that their solutions contain exponential functions of the independent variable.
- Exponential models describe population systems having infinite food resources.
- Population models may contain a migration term.
- The logistic equation is a population model with finite food resources.
- The interacting species model describes the interaction of two species with finite food resources.

1.2.1. Linear Growth and Decay.

Remarks:

- Linear models describe physical situations where a function changes in a linear way with respect to its independent variable.
- The models can be discrete or continuous.

Example 1 (Discrete Model): Consider a swimming pool that is initially empty. Every minute a bucket of K gallons is added to the pool. Write a mathematical model describing the amount of water W as function of the number n of minutes.

Solution:

$$W(n+1) - W(n) = K, \qquad W(0) = 0.$$

◁

Example 2 (Continuous Model): Consider a swimming pool that is initially empty. Water is added to the pool using a hose at a constant rate of K gallons per minute. Write a mathematical model describing the amount of water W as function of time t.

Solution:

$$W'(t) = K, \qquad W(0) = 0.$$

The solution is

$$W(t) = K\,t.$$

This is why these models are called linear growth models: the solution is a linear function of the independent variable t.

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Remarks:

- $K > 0$ describe water added to the pool.
- $K < 0$ describe water taken out of the pool.

1.2.2. Exponential Growth and Decay.

Remarks:

- Exponential models describe physical situations where a function changes in an exponential way with respect to its independent variable.
- Examples are population models with unlimited food resources.
- The models can be discrete or continuous.

Example 1 (Discrete Model): Consider a bacteria population in a Petri dish having **unlimited** food resources. Denote by $P(n)$ the bacteria population after n hours, where $P(0)$ is the initial bacteria population. The increment in the bacteria population every hour is equal to r times the amount of bacteria in the previous hour. Write a mathematical model describing the amount of bacteria P as function of the number n of hours.

Solution:

$$P(n+1) - P(n) = r\,P(n), \qquad P(0) \quad \text{is the initial population.}$$

$\lhd$

Example 2 (Continuous Model): Consider a bacteria population in a Petri dish having **unlimited** food resources. Denote by $P(t)$ the bacteria population at the time t, where $P(0)$ is the initial bacteria population. The rate of change in the bacteria population at the time t is equal to r times the amount of bacteria at that time. Write a mathematical model describing the amount of bacteria P as function of time t.

Solution:

$$P'(t) = r\,P(t), \qquad P(0) \quad \text{is the initial population.}$$

The solution is

$$\frac{P'}{P} = r \quad \Rightarrow \quad \ln(P(t)) = rt + c \quad \Rightarrow \quad P(t) = e^{rt+c} = e^{rt}\,e^{c} \quad \Rightarrow \quad P(0) = e^{c}.$$

$$P(t) = P(0)\,e^{rt}.$$

This is why these models are called exponential growth models: the solution is an exponential function of the independent variable t. $\lhd$

Remarks:

- r is the population rate change per capita.
- $r > 0$ describe exponential growth models.
- $r < 0$ describe exponential decay models.

Read in Lecture Notes:

- Radioactive half-life.
- Using radioactive decay to date remains.

1.2.3. Migration Terms.

Example 3 (Immigration): Describe a village population when they have **unlimited** food, the rate of population growth per capita is $r > 0$, and they have an immigration rate of K persons per unit time.

Solution:

$$P'(t) = r\,P(t) + K,$$

$$\frac{P'(t)}{r\,P(t)+K} = 1 \quad \Rightarrow \quad \int \frac{P'(t)}{r\,P(t)+K}\,dt = \int dt$$

$$\int \frac{dp}{rp+K} = t + c_0 \quad \Rightarrow \quad \frac{1}{r}\int \frac{dp}{p+K/r} = t + c_0$$

$$\ln|p + K/r| = rt + c_1 \quad \Rightarrow \quad |P(t) + K/r| = e^{rt+c_1} = e^{rt}\,e^{c_1}$$

$$P(t) + K/r = c_2\,e^{rt} \quad \Rightarrow \quad P(t) = c_2\,e^{rt} - K/r.$$

$$P(0) = c_2 - K/r \quad \Rightarrow \quad c_2 = P(0) + K/r \quad \Rightarrow \quad P(t) = \Big(P(0) + \frac{K}{r}\Big)\,e^{rt} - \frac{K}{r}.$$

$\triangleleft$

Remarks:

- K is the migration constant.
- When $K > 0$ the constant is called immigration.
- When $K < 0$ the constant is called emigration.

1.2.4. The Logistic Equation.

Remark: The logistic equation is a population model with finite food resources.

If the population $P(t)$ is small: $P'(t) \simeq r\,P(t) > 0$.

If the population $P(t)$ is large: $P'(t) < 0$.

Definition 1. The ***logistic equation*** for the function P, which depends on the independent variable t, is

$$P'(t) = r\,P(t)\Big(1 - \frac{P(t)}{P_c}\Big), \tag{1.2.1}$$

$r > 0$ is the growth constant and $P_c > 0$ is the carrying capacity.

Example 1: Suppose the function P is solution to the logistic equation

$$P'(t) = r\,P(t)\Big(1 - \frac{P(t)}{P_c}\Big).$$

(a) For what values of P is the population in equilibrium—that is, time independent?
(b) For what values of P is the population increasing in time?
(c) For what values of P is the population decreasing in time?

Solution:

(a) If P is an equilibrium solution, then P constant, so, $P' = 0$. The equation says

$$0 = \tilde{P}' = r\,\tilde{P}\Big(1 - \frac{\tilde{P}}{P_c}\Big) \quad \Rightarrow \quad \tilde{P} = 0 \quad \text{or} \quad \tilde{P} = P_c.$$

(b) If P is increasing, then $P' > 0$, then

$$P'(t) = r\,P(t)\Big(1 - \frac{P(t)}{P_c}\Big) > 0, \qquad r > 0, \quad P \geqslant 0$$

implies

$$\Big(1 - \frac{P(t)}{P_c}\Big) > 0 \quad \Rightarrow \quad 0 < P(t) < P_c.$$

(c) If P is decreasing, then $P' < 0$, but

$$\Big(1 - \frac{P(t)}{P_c}\Big) < 0 \quad \Rightarrow \quad P(t) > P_c,$$

since P cannot be negative.

Discuss the meaning of the carrying capacity.

$\triangleleft$

1.2.5. Interacting Species.

Problem: Write a simple model to describe how rabbits and sheep populations evolve in time when they **compete** on **finite** food resources on a particular piece of land.

Solution:

- Suppose the species do not interact. R are rabbits and S are sheep.

$$R' = r_r\, R\Big(1 - \frac{R}{R_c}\Big)$$
$$S' = r_s\, S\Big(1 - \frac{S}{S_c}\Big),$$

  where r_r, r_s are the growth rates and R_c, S_c are the carrying capacities.

- Introduce the effect or sheep on rabbits.

$$R' = r_r\, R\Big(1 - \frac{R}{R_c}\Big) - c_1\, R\, S, \qquad c_1 > 0.$$

  The product measures the encounters on the field.

- Introduce the effect of or rabbits on sheep.

$$S' = r_s\, S\Big(1 - \frac{S}{S_c}\Big) - c_2\, R\, S, \qquad c_2 > 0.$$

  The product measures the encounters on the field.

$\lhd$

Definition 2. The ***interacting species equation*** for the functions x and y, which depend on the independent variable t, are

$$x' = r_x\, x\Big(1 - \frac{x}{x_c}\Big) + \alpha\, x\, y$$
$$y' = r_y\, y\Big(1 - \frac{y}{y_c}\Big) + \beta\, x\, y,$$

where the constants r_x, r_y and x_c, x_c are positive and α, β are real numbers.

Example 1.2.1. The following systems are models of the populations of pair of species that either **compete** for resources (an increase in one species decreases the growth rate in the other) or **cooperate** (an increase in one species increases the growth rate in the other). For each of the following systems identify the independent and dependent variables, the parameters, such as growth rates, carrying capacities, measure of interactions between species. Do the species compete of cooperate?

(a)

$$\frac{dx}{dt} = c_1 x - c_1 \frac{x^2}{K_1} - b_1\, xy$$
$$\frac{dy}{dt} = c_2 y - c_2 \frac{y^2}{K_2} - b_2\, xy.$$

(b)

$$\frac{dx}{dt} = x - \frac{x^2}{5} + 5\, xy$$
$$\frac{dy}{dt} = 2\, y - \frac{y^2}{6} + 2\, xy.$$

Solution:

(a)

- The species compete.
- t is the independent variable.
- x, y are the dependent variables.
- c_1, c_2 are the growth rates.
- K_1, K_2 are the carrying capacities.
- b_1, b_2 are the competition coefficients.

(b)

- The species coperate.
- t is the independent variable.
- x, y are the dependent variables.
- 1, 2 are the growth rates.
- 5, 12 (not 6) are the carrying capacities.
- 5, 2 are the competition coefficients.

Question: If x are elephants and y are chipmunks, then is $b_1 > b_2$ or $b_2 > b_1$?

Answer: $b_2 > b_1$.

$\triangleleft$

1.3. Qualitative Analysis

Section Objective(s):

- The Existence of Solutions Theorem.
- Direction Fields.
- Autonomous Equations.

Remarks:

- If the equation is nice enough, then there are solutions.
- However, there is no explicit formula for the solutions of all differential equations.
- The simple functions we know are not enough to write their solutions.
- Simple functions are power, rational functions exponentials, logs, trigonometric functions
- There are more equations than simple functions needed to write their solutions.
- It is important to study qualitative methods to describe solutions to differential equations.
- We get information about the solutions of differential equations without solving the equation.
  (a) Direction Fields Method, works with all equations.
  (b) Autonomous Equations Method, works with particular equations.

1.3.1. The Existence of Solutions Theorem.

Theorem 1.3.1. (Picard-Lindelöf) Consider the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0.$$

If the function f and its partial derivative $\partial_y f$ are continuous on some rectangle on the ty-plane containing the point (t_0, y_0) in its interior,

then there is a unique solution y ________________ of the initial value problem above on an open interval I containing the point t_0.

Remarks:

(1) An initial value problem ____________ means to find a solution to both _____ a differential equation and an initial condition.

(2) There is no formula ________ for the solution in this Theorem.

(3) Results with no formula ________ are still useful ______

Example 1.3.1. Determine whether the functions y_1 and y_2 given by their graphs in Fig. 2 can be solutions of the same differential equation satisfying the hypotheses in the Picard-Lindelöf Theorem.

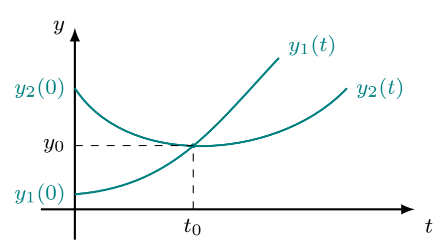

FIGURE 2. The graph of two functions.

Solution:

- No.
- Solution graphs do not intersect.
- If they did, at (t_0, y_0) the IVP would have two solutions;
- But the Theorem above says IVP always have only one solution.
- So no intersections.

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1.3.2. Direction Fields.

Remark: We interpret $f(t, y)$ at each point (t, y) on the ty-plane as

the value of a slope of a segment.

Definition 1.6.3. The ***direction field*** of the differential equation

$$y'(t) = f(t, y(t))$$

is the graph on the ty-plane of $f(t, y)$

as slopes of segments.

Example 1.6.11: Find the direction field of the equation $y' = \sin(y)$, and sketch a few solutions to the differential equation for different initial conditions.

Solution: We first mention that the equation above can be solved and the solutions are

$$\frac{\sin(y)}{\big(1+\cos(y)\big)} = \frac{\sin(y_0)}{\big(1+\cos(y_0)\big)}\, e^t.$$

for any $y_0 \in \mathbb{R}$. This is an equation that defines the solution function y. There are no derivatives in the equation, so this is not a differential equation; We call it an algebraic equation. However, the graphs of these solutions are not simple to do. But the direction field is simple to plot and it can be seen in Fig. 3. From that direction field one can see what the graph of the solutions look like. ◁

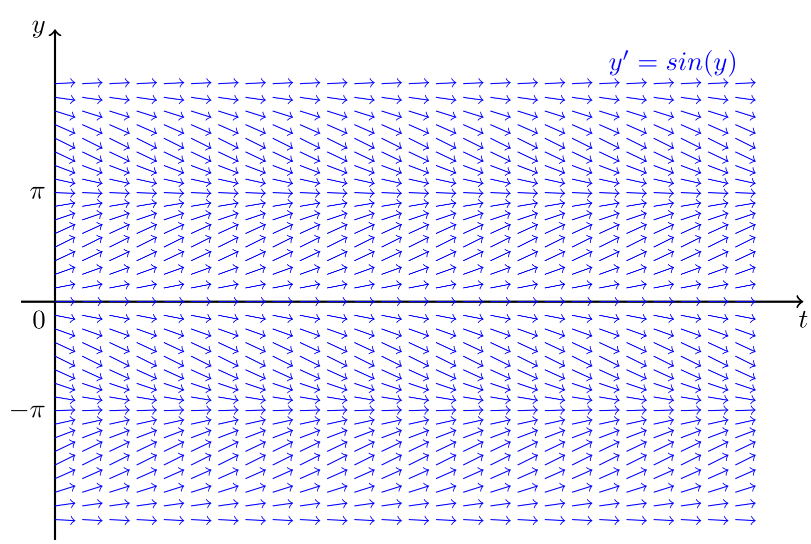

FIGURE 3. Direction field for the equation $y' = \sin(y)$.

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1.3.3. Autonomous Equations.

Definition 6.1.1. A first order equation is ***autonomous*** iff

$$y' = f(\cancel{t}, y),$$

where $y' = \dfrac{dy}{dt}$, and the function f does not depend explicitly on t.

Remark: An important example of an autonomous equation is

the logistic equation.

$$P' = rP\left(1 - \frac{P}{P_c}\right), \qquad r > 0, \qquad P_c > 0.$$

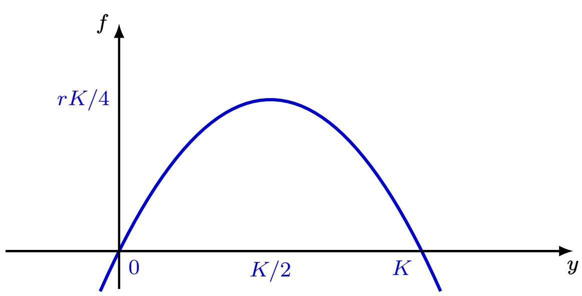
Remark: The logistic equation can be solved exactly.

$$P(t) = \frac{P_c\, P_0}{P_0 + (P_c - P_0)\, e^{-rt}}, \qquad P(0) = P_0.$$

Example 6.1.7: Sketch a qualitative graph of solutions of

$$y' = ry\left(1 - \frac{y}{K}\right), \qquad y(0) = y_0, \qquad r > 0, \qquad K > 0.$$

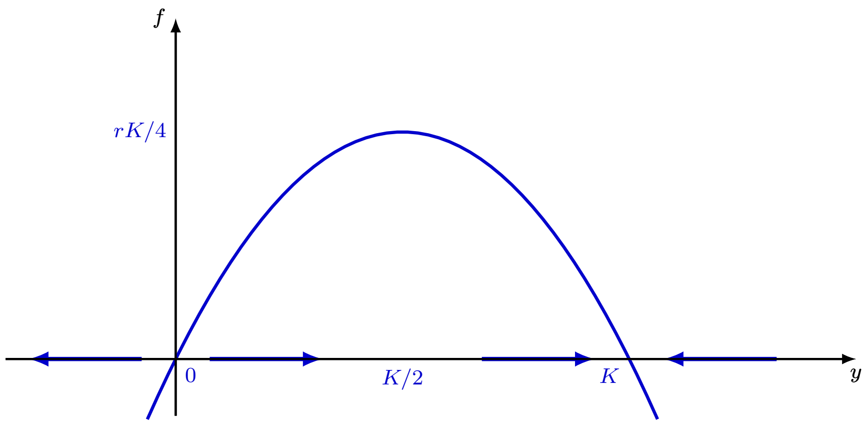
Solution:
(1) Graph $f(y) = ry(1 - y/K)$



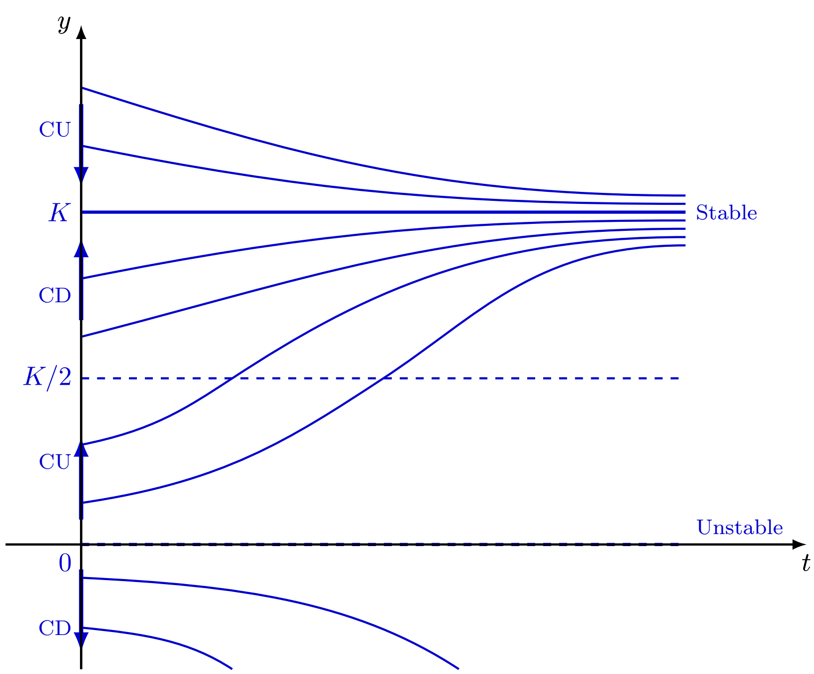
(2) Find the critical points: y_c is a critical point iff $f(y_c) = 0$

$$f(y) = ry(1 - y/K) \quad \Rightarrow \quad y_0 = 0, \quad y_1 = K.$$

(3) Find the increasing-decreasing intervals of f.



(4) We can skip the concavity regions.
(5) Move the horizontal y-axis into a vertical axis, and add a horizontal t-axis.



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1.4. Separable Equations

Section Objective(s):

- Separable Differential Equations
- Euler Homogeneous Equations
- Solving Euler Homogeneous Equations

Remarks:

- Separable differential equations are simple to solve.
- Integrate on both sides just works.
- Euler homogeneous equations are not separable.
- Euler homogeneous equations can be transformed into separable equations.
- One then solves the separable equation and then transforms back the solution.

1.4.1. Separable Differential Equations.

Definition 1. A ***separable*** differential equation for the function y is

$$h(y)\, y' = g(t),$$

where h, g ____ are given functions.

Remark:

$$h(y)\, y' = g(y)$$

- The left-hand side depends explicitly only on y, so any t dependence is through y.
- The right-hand side depends only on t.
- And the left-hand side is of the form (something on y) $\times\, y'$.

Example 1.4.1. Find all solutions y to the differential equation

$$-\frac{y'}{y^2} = \cos(2t).$$

Solution: The differential equation above is separable, with

$$g(t) = \cos(2t), \qquad h(y) = -\frac{1}{y^2},$$

therefore, it can be integrated as follows:

$$-\frac{y'(t)}{y^2(t)} = \cos(2t) \quad \Leftrightarrow \quad \int -\frac{y'(t)}{y^2(t)}\, dt = \int \cos(2t)\, dt + c.$$

Again the substitution

$$y = y(t), \qquad dy = y'(t)\, dt$$

implies that

$$\int -\frac{dy}{y^2} = \int \cos(2t)\, dt + c \quad \Leftrightarrow \quad \frac{1}{y} = \frac{1}{2}\sin(2t) + c.$$

So, we get the implicit and explicit form of the solution,

$$\frac{1}{y(t)} = \frac{1}{2}\sin(2t) + c \quad \Leftrightarrow \quad y(t) = \frac{2}{\sin(2t) + 2c}.$$

$\triangleleft$

Remark: Let's find the general rule for the solution formula:

$$-\frac{1}{y^2}\,y' = \cos(2t) \quad \Rightarrow \quad \frac{1}{y} = \frac{1}{2}\sin(2t) + c$$

$$h(y)\,y' = g(t), \quad \Rightarrow \quad H(y) = G(t) + c$$

$$h(y) = -\frac{1}{y^2}, \quad \Rightarrow \quad H(y) = \frac{1}{y}$$

$$g(t) = \cos(2t), \quad \Rightarrow \quad G(t) = \frac{1}{2}\sin(2t).$$

where H is an antiderivative of h, that is, $H(y) = \int h(y)\,dy$.

and G is an antiderivative of g, that is, $G(t) = \int g(t)\,dt$.

Theorem 1. (Separable Equations) If h, g are continuous, with $h \neq 0$, then

$$h(y)\,y' = g(t)$$

has infinitely many solutions y satisfying the algebraic equation

$$H(y(t)) = G(t) + c, \qquad c \in \mathbb{R},$$

where H and G are antiderivatives of h, and g.

Remark: An antiderivative of $h(y)$ is $H(y) = \int h(y)\,dy$, and an antiderivative of $g(t)$ is given by $G(t) = \int g(t)\,dt$.

1.4.2. Euler Homogeneous Equations.

Definition 2. An ***Euler homogeneous*** differential equation has the form

$$y'(t) = F\Big(\frac{y(t)}{t}\Big).$$

Example 1.4.2.

$$(1)\quad y' = \frac{3 + 2\,(y/t)^3}{(y/t)}, \qquad (2)\quad y' = \frac{\cos(y)}{2t}, \qquad (3)\quad y' = \frac{t^2 + 3y^2}{2ty},$$

(1) This is Euler Homogeneous.

(2) This is Not Euler Homogeneous.

(3) This is Euler homogeneous.

Example 1.4.3. Show that $y' = \dfrac{t^2 + 3y^2}{2ty}$ is Euler Homogeneous.

Solution:

$$y' = \frac{(t^2 + 3y^2)}{2ty} = \frac{(t^2 + 3y^2)}{2ty}\,\frac{\left(\frac{1}{t^2}\right)}{\left(\frac{1}{t^2}\right)} \quad\Rightarrow\quad y' = \frac{1 + 3\left(\frac{y}{t}\right)^2}{2\left(\frac{y}{t}\right)}.$$

Theorem 2. If there is an integer n such that

$$p(ct, cy) = c^n\, p(t, y), \qquad q(ct, cy) = c^n\, q(t, y), \qquad \text{for all constant } c > 0,$$

then $y' = \dfrac{p(t,y)}{q(t,y)}$ is Euler homogeneous.

Proof: Choose $c = \dfrac{1}{t}$, then

$$\frac{p(t,y)}{q(t,y)} = \frac{p(t,y)}{q(t,y)}\,\frac{1/t^n}{1/t^n} = \frac{p(t/t, y/t)}{q(t/t, y/t)} = \frac{p(1/y/t)}{q(1, y/t)} = F(y/t).$$

$\square$

1.4.3. Solving Euler Homogeneous Equations.

Theorem 3. The Euler homogeneous equation

$$y' = F\Big(\frac{y}{t}\Big)$$

for the function y determines a separable equation for $v = \dfrac{y}{t}$, given by

$$\frac{v'}{\big(F(v) - v\big)} = \frac{1}{t}.$$

Proof: If $y' = f(t, y)$ is Euler homogeneous, then we known that it can be written as $y' = F(y/t)$, where $F(y/t) = f(1, y/t)$. Introduce the function $v = y/t$ into the differential equation,

$$y' = F(v).$$

We still need to replace y' in terms of v. This is done as follows,

$$y(t) = t\,v(t) \quad \Rightarrow \quad y'(t) = v(t) + t\,v'(t).$$

Introducing these expressions into the differential equation for y we get

$$v + t\,v' = F(v) \quad \Rightarrow \quad v' = \frac{\big(F(v) - v\big)}{t} \quad \Rightarrow \quad \frac{v'}{\big(F(v) - v\big)} = \frac{1}{t}.$$

The equation on the far right is separable. This establishes the Theorem.

$\square$

Example 1.4.4. Find all solutions y of the differential equation $y' = \dfrac{t^2 + 3y^2}{2ty}$.

Solution: The equation is Euler homogeneous, since

$$f(ct, cy) = \frac{c^2t^2 + 3c^2y^2}{2(ct)(cy)} = \frac{c^2(t^2 + 3y^2)}{c^2(2ty)} = \frac{t^2 + 3y^2}{2ty} = f(t, y).$$

Next we compute the function F. Since the numerator and denominator are homogeneous degree "2" we multiply the right-hand side of the equation by "1" in the form $(1/t^2)/(1/t^2)$,

$$y' = \frac{(t^2 + 3y^2)}{2ty} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{1}{t^2}\right)} \quad \Rightarrow \quad y' = \frac{1 + 3\left(\frac{y}{t}\right)^2}{2\left(\frac{y}{t}\right)}.$$

Now we introduce the change of functions $v = y/t$,

$$y' = \frac{1 + 3v^2}{2v}.$$

Since $y = t\,v$, then $y' = v + t\,v'$, which implies

$$v + t\,v' = \frac{1 + 3v^2}{2v} \quad \Rightarrow \quad t\,v' = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v} = \frac{1 + v^2}{2v}.$$

We obtained the separable equation

$$v' = \frac{1}{t}\left(\frac{1 + v^2}{2v}\right).$$

We rewrite and integrate it,

$$\frac{2v}{1 + v^2}\, v' = \frac{1}{t} \quad \Rightarrow \quad \int \frac{2v}{1 + v^2}\, v'\, dt = \int \frac{1}{t}\, dt + c_0.$$

The substitution $u = 1 + v^2(t)$ implies $du = 2v(t)\, v'(t)\, dt$, so

$$\int \frac{du}{u} = \int \frac{dt}{t} + c_0 \quad \Rightarrow \quad \ln(u) = \ln(t) + c_0 \quad \Rightarrow \quad u = e^{\ln(t) + c_0}.$$

But $u = e^{\ln(t)} e^{c_0}$, so denoting $c_1 = e^{c_0}$, then $u = c_1 t$. So, we get

$$1 + v^2 = c_1 t \quad \Rightarrow \quad 1 + \left(\frac{y}{t}\right)^2 = c_1 t \quad \Rightarrow \quad y(t) = \pm t\sqrt{c_1 t - 1}.$$

$\triangleleft$

1.5. Linear Equations

Section Objective(s):
- Constant Coefficient Equations.
- Variable Coefficient Equations.
- The Integrating Factor Method.

Remarks:

- The study equations of the form $y' = a(t)\,y + b(t)$.
- Constant coefficients linear equations are separable .
- We review how to solve these equations.
- Variable coefficients linear equations may not be separable .
- And integrating on both sides of the equation actually does not work .
- A new idea is needed to solve variable coefficients equations.
- The new idea is to transform the linear equation into a total derivative .

$$y' = a(t)\,y + b(t) \quad \longrightarrow \quad \big(\,\psi(t, y(t))\,\big)' = 0.$$

- This is what integrating factor method does .

1.5.1. Linear Constant Coefficient Equations.

Definition 1. A ***linear differential equation*** on the function y is

$$y' = a(t)\, y + b(t)$$

The equation has **constant** coefficients if both a and b are constants, otherwise the equation has **variable** coefficients.

Example 1. (Constant Coefficients): Solve linear constant coefficients equations using that they are separable equations.

We wrote the equation $y' = a\,y + b$ as follows $y' = a\left(y + \frac{b}{a}\right)$. The critical step was the following: since b/a is constant, then $(b/a)' = 0$, hence

$$\Big(y + \frac{b}{a}\Big)' = a\,\Big(y + \frac{b}{a}\Big).$$

At this point the equation was simple to solve,

$$\frac{(y + \frac{b}{a})'}{(y + \frac{a}{b})} = a \quad \Rightarrow \quad \ln\Big(\Big|y + \frac{b}{a}\Big|\Big)' = a \quad \Rightarrow \quad \ln\Big(\Big|y + \frac{b}{a}\Big|\Big) = c_0 + at.$$

We now computed the exponential on both sides, to get

$$\Big|y + \frac{b}{a}\Big| = e^{c_0 + at} = e^{c_0}\, e^{at} \quad \Rightarrow \quad y + \frac{b}{a} = (\pm e^{c_0})\, e^{at},$$

and calling $c = \pm e^{c_0}$ we got the formula $y(t) = c\, e^{at} - \frac{b}{a}$,

$\lhd$

Example 2. (Variable Coefficients with $b = 0$): Solve linear variable coefficients equations, $y' = a(t)\,y$, using that they are separable equations.

Solution:

$$\frac{y'}{y} = a(t) \quad \Rightarrow \quad \ln(|y|)' = a(t) \quad \Rightarrow \quad \ln(|y(t)|) = A(t) + c_0,$$

where $A = \int a\,dt$, is a primitive or antiderivative of a. Therefore,

$$y(t) = \pm e^{A(t)+c_0} = \pm e^{A(t)}\,e^{c_0} \quad \Rightarrow \quad y(t) = c\,e^{A(t)}, \qquad c = \pm e^{c_0}.$$

$\lhd$

Example 3.: Find all solutions of The solutions of $y' = 2t\,y$.

Solution:
$y(t) = c\,e^{t^2}$, where $c \in \mathbb{R}$.

$\lhd$

Remark: The case b/a non-constant cannot be solved with this idea.

1.5.2. Variable Coefficient Equations.

Theorem 1. (Variable Coefficients) If the functions a, b are continuous, then

$$y' = a(t)\, y + b(t), \tag{1.5.1}$$

has infinitely many solutions given by

$$y(t) = c\, e^{A(t)} + e^{A(t)} \int e^{-A(t)}\, b(t)\, dt, \tag{1.5.2}$$

where $A(t) = \int a(t)\, dt$ and $c \in \mathbb{R}$.

Remarks:

(a) The expression in Eq. (1.5.2) is called the general solution.

(b) We solve these equations using the integrating factor method.

(c) The function $\mu(t) = e^{-A(t)}$ is the integrating factor.

Example 4. (Integrating Factor Method): Find all the solutions of the equation

$$t y' = -2y + 4t^2, \quad \text{with} \quad t > 0.$$

Solution: Rewrite the equation as

$$y' = -\frac{2}{t}\, y + 4t \quad \Leftrightarrow \quad a(t) = -\frac{2}{t}, \quad b(t) = 4t. \tag{1.5.3}$$

Rewrite again,

$$y' + \frac{2}{t}\, y = 4t.$$

Multiply by a function μ,

$$\mu\, y' + \frac{2}{t}\, \mu\, y = \mu\, 4t.$$

Choose μ solution of

$$\frac{2}{t}\, \mu = \mu' \quad \Rightarrow \quad \ln(|\mu|)' = \frac{2}{t} \quad \Rightarrow \quad \ln(|\mu|) = 2\ln(t) = \ln(t^2) \quad \Rightarrow \quad \mu(t) = \pm t^2.$$

We choose $\mu = t^2$. Multiply the differential equation by this μ,

$$t^2\, y' + 2t\, y = 4t\, t^2 \quad \Rightarrow \quad (t^2\, y)' = 4t^3.$$

If we write the right-hand side also as a derivative,

$$\big(t^2\, y\big)' = \big(t^4\big)' \quad \Rightarrow \quad \big(t^2\, y - t^4\big)' = 0.$$

So a potential function is $\psi(t, y(t)) = t^2\, y(t) - t^4$. Integrating on both sides we obtain

$$t^2\, y - t^4 = c \quad \Rightarrow \quad t^2\, y = c + t^4 \quad \Rightarrow \quad y(t) = \frac{c}{t^2} + t^2.$$

$\lhd$

Example 5. (Initial Value Problem): Find the solution to the initial value problem

$$ty' + 2y = 4t^2, \qquad t > 0, \qquad y(1) = 2.$$

Solution: The general solution is $y(t) = \dfrac{c}{t^2} + t^2$. The initial condition implies that

$$2 = y(1) = c + 1 \quad \Rightarrow \quad c = 1 \quad \Rightarrow \quad y(t) = \frac{1}{t^2} + t^2.$$

$\triangleleft$

Example 6. (Extra Example 1): Find all solutions to the differential equation

$$y' = \frac{3}{t}y + t^5, \quad \text{for} \quad t > 0.$$

Solution: Rewrite the equation with y on only one side,

$$y' - \frac{3}{t}\,y = t^5.$$

Multiply the differential equation by a function μ, which we determine later,

$$\mu(t)\Big(y' - \frac{3}{t}\,y\Big) = t^5\,\mu(t) \quad \Rightarrow \quad \mu(t)\,y' - \frac{3}{t}\,\mu(t)\,y = t^5\,\mu(t).$$

We need to choose a positive function μ having the following property,

$$-\frac{3}{t}\,\mu(t) = \mu'(t) \quad \Rightarrow \quad -\frac{3}{t} = \frac{\mu'(t)}{\mu(t)} \quad \Rightarrow \quad -\frac{3}{t} = \big(\ln(|\mu|)\big)'$$

Integrating,

$$\ln(|\mu|) = -\int \frac{3}{t}\,dt = -3\,\ln(|t|) + c_0 = \ln(|t|^{-3}) + c_0 \quad \Rightarrow \quad \mu = (\pm e^{c_0})\,e^{\ln(|t|^{-3})},$$

so we get $\mu = (\pm e^{c_0})\,|t|^{-3}$. We need only one integrating factor, so we choose $\mu = t^{-3}$. We now go back to the differential equation for y and we multiply it by this integrating factor,

$$t^{-3}\Big(y' - \frac{3}{t}\,y\Big) = t^{-3}\,t^5 \quad \Rightarrow \quad t^{-3}\,y' - 3\,t^{-4}\,y = t^2.$$

Using that $-3\,t^{-4} = (t^{-3})'$ and $t^2 = \Big(\dfrac{t^3}{3}\Big)'$, we get

$$t^{-3}\,y' + (t^{-3})'\,y = \Big(\frac{t^3}{3}\Big)' \quad \Rightarrow \quad \big(t^{-3}\,y\big)' = \Big(\frac{t^3}{3}\Big)' \quad \Rightarrow \quad \Big(t^{-3}\,y - \frac{t^3}{3}\Big)' = 0.$$

This last equation is a total derivative of a potential function $\psi(t,y) = t^{-3}\,y - \dfrac{t^3}{3}$. Since the equation is a total derivative, this confirms that we got a correct integrating factor. Now we need to integrate the total derivative, which is simple to do,

$$t^{-3}\,y - \frac{t^3}{3} = c \quad \Rightarrow \quad t^{-3}\,y = c + \frac{t^3}{3} \quad \Rightarrow \quad y(t) = c\,t^3 + \frac{t^6}{3},$$

where c is an arbitrary constant.

$\lhd$

Example 7. (Extra Example 2): Find the solution of

$$ty' = 2y + 4t^3\cos(4t), \qquad y\left(\frac{\pi}{8}\right) = 0.$$

Solution: Rewrite the equation as

$$y' - \frac{2}{t}\,y = 4t^2\cos(4t) \quad \Leftrightarrow \quad a(t) = \frac{2}{t}, \quad b(t) = 4t^2\cos(4t). \tag{1.5.4}$$

Multiply by a function μ,

$$\mu\, y' - \frac{2}{t}\,\mu\, y = \mu\, 4t^2\cos(4t).$$

Choose μ solution of

$$-\frac{2}{t}\,\mu = \mu' \quad \Rightarrow \quad \ln(|\mu|)' = -\frac{2}{t} \quad \Rightarrow \quad \ln(|\mu|) = -2\ln(t) = \ln(t^{-2}) \quad \Rightarrow \quad \mu(t) = \pm\frac{1}{t^2}.$$

We choose $\mu = \frac{1}{t^2}$. Multiply the differential equation by this μ,

$$\frac{y'}{t^2} - \frac{2y}{t^3} = 4\cos(4t) \quad \Rightarrow \quad \Big(\frac{y}{t^2}\Big)' = 4\cos(4t).$$

If we write the right-hand side also as a derivative,

$$\Big(\frac{y}{t^2}\Big)' = \big(\sin(4t)\big)' \quad \Rightarrow \quad \Big(\frac{y}{t^2} - \sin(4t)\Big)' = 0.$$

So a potential function is $\psi(t, y(t)) = \frac{y}{t^2} - \sin(4t)$. Integrating on both sides we obtain

$$\frac{y}{t^2} - \sin(4t) = c \quad \Rightarrow \quad y(t) = ct^2 + t^2\sin(4t).$$

Using the initial condition, we find $c = -1$, so the solution to the IVP is

$$y(t) = -t^2 + t^2\sin(4t).$$

$\triangleleft$

1.6. Approximate Solutions

Section Objective(s):
- The Existence of Solutions Theorem.
- The Picard Iteration.
- Linear vs Nonlinear Equations.

Remarks:

- If the equation is nice enough, then there are solutions.
- The theorem is proved using the Picard iteration.
- The Picard iteration creates a sequence of functions.
- The solution of the equation is the limit of the sequence.
- We compare results on solutions of linear and nonlinear equations.

1.6.1. The Existence of Solutions Theorem.

Theorem 1.3.1. (Picard-Lindelöf) Consider the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0.$$

If the function f and its partial derivative $\partial_y f$ are continuous on some rectangle on the ty-plane containing the point (t_0, y_0) in its interior,

then there is a unique solution y ____________________ of the initial value problem above on a smaller rectangle containing the condition (t_0, y_0).

Idea of the Proof: The Picard Iteration.

(a) Transform the differential equation into an integral equation:

$$\int_{t_0}^{t} y'(s)\,ds = \int_{t_0}^{t} f(s, y(s))\,ds \quad \Rightarrow \quad y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\,ds,$$

where we have used the Fundamental Theorem of Calculus on the left-hand side of the first equation to get the second equation.

(b) Introduce a sequence of functions, called approximate solutions, as follows:

$$y_0(t) = y(t_0),$$
$$y_{n+1}(t) = y(t_0) + \int_{t_0}^{t} f(s, y_n(s))\,ds, \quad n \geqslant 0.$$

Remark: One can show that $\lim_{n\to\infty} y_n(t) = y(t)$ exists and this limit satisfies

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\,ds$$

and y is differentiable, so it also satisfies $y' = f(t, y)$.

□

1.6.2. The Picard Iteration.

Example 1: Use three iterations of Picard's iteration procedure to find and approximate solution to

$$y' = 2\,y + 3 \qquad y(0) = 1.$$

Remark: We can compute the solution using the integrating factor method.

$$e^{-2t}\,(y' - 2\,y) = e^{-2t}\,3 \quad \Rightarrow \quad e^{-2t}\,y = -\frac{3}{2}\,e^{-2t} + c \quad \Rightarrow \quad y(t) = c\,e^{2t} - \frac{3}{2};$$

and the initial condition implies

$$1 = y(0) = c - \frac{3}{2} \quad \Rightarrow \quad c = \frac{5}{2} \quad \Rightarrow \quad y(t) = \frac{5}{2}\,e^{2t} - \frac{3}{2}.$$

Solution: We first transform the differential equation into an integral equation.

$$\int_0^t y'(s)\,ds = \int_0^t (2\,y(s) + 3)\,ds \quad \Rightarrow \quad y(t) - y(0) = \int_0^t (2\,y(s) + 3)\,ds.$$

Using the initial condition, $y(0) = 1$,

$$y(t) = 1 + \int_0^t (2\,y(s) + 3)\,ds.$$

We now define the sequence of approximate solutions:

$$y_0 = y(0) = 1, \quad y_{n+1}(t) = 1 + \int_0^t (2\,y_n(s) + 3)\,ds, \quad n \geqslant 0.$$

We now compute the first elements in the sequence. We said $y_0 = 1$, now y_1 is given by

$$n = 0, \quad y_1(t) = 1 + \int_0^t (2\,y_0(s) + 3)\,ds = 1 + \int_0^t 5\,ds = 1 + 5t.$$

So $y_1 = 1 + 5t$. Now we compute y_2,

$$y_2 = 1 + \int_0^t (2\,y_1(s) + 3)\,ds = 1 + \int_0^t \big(2(1+5s) + 3\big)\,ds \quad \Rightarrow \quad y_2 = 1 + \int_0^t \big(5 + 10s\big)\,ds = 1 + 5t + 5t^2.$$

So we've got $y_2(t) = 1 + 5t + 5t^2$. Now y_3,

$$y_3 = 1 + \int_0^t (2\,y_2(s) + 3)\,ds = 1 + \int_0^t \big(2(1 + 5s + 5s^2) + 3\big)\,ds$$

so we have,

$$y_3 = 1 + \int_0^t \big(5 + 10s + 10s^2\big)\,ds = 1 + 5t + 5t^2 + \frac{10}{3}\,t^3.$$

So we obtained $y_3(t) = 1 + 5t + 5t^2 + \dfrac{10}{3}\,t^3$. We now rewrite this expression so we can get a power series expansion that can be written in terms of simple functions. The first step is done already, to write the powers of t as t^n, for $n = 1, 2, 3$,

$$y_3(t) = 1 + 5t^1 + 5t^2 + \frac{5(2)}{3}\,t^3$$

We now multiply by one each term so we get the factorials $n!$ on each term

$$y_3(t) = 1 + 5\,\frac{t^1}{1!} + 5(2)\,\frac{t^2}{2!} + 5(2^2)\,\frac{t^3}{3!}$$

We then realize that we can rewrite the expression above in terms of power of $(2t)$, that is,

$$y_3(t) = 1 + \frac{5}{2}\,\frac{(2t)^1}{1!} + \frac{5}{2}\,\frac{(2t)^2}{2!} + \frac{5}{2}\,\frac{(2t)^3}{3!} = 1 + \frac{5}{2}\Big((2t) + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!}\Big).$$

From this last expressions simple to guess the n-th approximation

$$y_N(t) = 1 + \frac{5}{2}\Big((2t) + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots + \frac{(2t)^N}{N!}\Big) = 1 + \frac{5}{2}\sum_{k=1}^{N}\frac{(2t)^k}{k!}.$$

Recall now that the power series expansion for the exponential

$$e^{at} = \sum_{k=0}^{\infty}\frac{(at)^k}{k!} = 1 + \sum_{k=1}^{\infty}\frac{(at)^k}{k!} \quad \Rightarrow \quad \sum_{k=1}^{\infty}\frac{(at)^k}{k!} = (e^{at} - 1).$$

Then, the limit $N \to \infty$ is given by

$$y(t) = \lim_{N\to\infty} y_N(t) = 1 + \frac{5}{2}\sum_{k=1}^{\infty}\frac{(2t)^k}{k!} = 1 + \frac{5}{2}\big(e^{2t} - 1\big),$$

One last rewriting of the solution and we obtain

$$y(t) = \frac{5}{2}e^{2t} - \frac{3}{2}.$$

$\triangleleft$

1.6.3. Linear vs Nonlinear Equations.

Recall: The main theorem about solutions of linear equations.

Theorem 1. Given continuous functions a, b with domain (t_1, t_2), and constants $t_0 \in (t_1, t_2)$, $y_0 \in \mathbb{R}$, then the initial value problem

$$y' = a(t)\, y + b(t), \qquad y(t_0) = y_0,$$

has the unique solution on the domain (t_1, t_2), given by

$$y(t) = y_0 e^{A(t)} + e^{A(t)} \int_{t_0}^{t} e^{-A(s)}\, b(s)\, ds,$$

where $A(t) = \int_{t_0}^{t} a(s)\, ds$.

Solutions to linear equations satisfy:

(a) There is an explicit formula for all solutions.

(b) For every initial condition y_0 there is a unique solution.

(c) For every IC y_0 the domain of $y(t)$ is fixed, (t_1, t_2).

Solutions to nonlinear equations satisfy:

(1) There is no explicit formula for the solution of every nonlinear differential equation.

(2) Given an initial condition (t_0, y_0) there may be more than one solution.

(3) Given an initial condition (t_0, y_0) the domain of the solution $y(t)$ may change with y_0.

Example 2. (Linear vs. Non-Linear ODEs): The solutions of the following equations are examples of the properties above. Identify which example corresponds to which property and explain your reasoning.

(1) $y'(t) = \dfrac{t^2}{\big(y^4(t) + 8\, y^3(t) + 9\, y^2(t) + 6\, y(t) + 7\big)}$.

(2) $y'(t) = y^{1/3}(t), \qquad y(0) = 0.$

(3) $y'(t) = y^2(t), \qquad y(0) = y_0.$

$\lhd$

Example 3. (Extendability of solutions to linear equations): In the initial value problems below find the **maximum domain** where the solution is **certain** to exist.

(1) $t(t-5)\,y' = y, \qquad y(-1) = 4$

(2) $(t^2-4)\,y' - 5\ln(t)\,y = 3t, \qquad y(1) = 2$

Solution:

(1) The equation is $y' = \dfrac{y}{t(t-5)}$, so the equation is defined on

$$(-\infty, 0) \cup (0, 5) \cup (5, \infty).$$

The initial condition is at $t = -1$, so the interval where the equation is defined and contains the initial condition is

$$I = (-\infty, 0).$$

(2) The equation is $y' = \dfrac{5\,\ln(t)\,y}{(t^2-4)} + \dfrac{3t}{(t^2-4)}$, so the equation is defined on

$$(0, 2) \cup (2, \infty).$$

The initial condition is at $t = 1$, so the interval where the equation is defined and contains the initial condition is

$$I = (0, 2).$$

$\triangleleft$

Example 4. (Extra Example): Use three iterations of Picard's iteration procedure to find an approximate solution of

$$y' = 5t\,y, \qquad y(0) = 1.$$

Solution: We first transform the differential equation into an integral equation.

$$\int_0^t y'(s)\,ds = \int_0^t 5s\,y(s)\,ds \quad \Rightarrow \quad y(t) - y(0) = \int_0^t 5s\,y(s)\,ds.$$

Using the initial condition, $y(0) = 1$,

$$y(t) = 1 + \int_0^t 5s\,y(s)\,ds.$$

We now define the sequence of approximate solutions:

$$y_0 = y(0) = 1, \quad y_{n+1}(t) = 1 + \int_0^t 5s\,y_n(s)\,ds, \quad n \geqslant 0.$$

We now compute the first four elements in the sequence. The first one is $y_0 = y(0) = 1$, the second one y_1 is given by

$$n = 0, \quad y_1(t) = 1 + \int_0^t 5s\,ds = 1 + \frac{5}{2}\,t^2.$$

So $y_1 = 1 + (5/2)t^2$. Now we compute y_2,

$$\begin{aligned} y_2 &= 1 + \int_0^t 5s\,y_1(s)\,ds \\ &= 1 + \int_0^t 5s\,\big(1 + \frac{5}{2}\,s^2\big)\,ds \\ &= 1 + \int_0^t \big(5s + \frac{5^2}{2}\,s^3\big)\,ds \\ &= 1 + \frac{5}{2}\,t^2 + \frac{5^2}{8}\,t^4. \end{aligned}$$

So we obtained $y_2(t) = 1 + \dfrac{5}{2}t^2 + \dfrac{5^2}{2^3}\,t^4$. A similar calculation gives us y_3,

$$\begin{aligned} y_3 &= 1 + \int_0^t 5s\,y_2(s)\,ds \\ &= 1 + \int_0^t 5s\,\big(1 + \frac{5}{2}\,s^2 + \frac{5^2}{2^3}\,s^4\big)\,ds \end{aligned}$$

$$
\begin{aligned}
y_3 &= 1 + \int_0^t \Big(5s + \frac{5^2}{2}\, s^3 + \frac{5^3}{2^3}\, s^5\Big)\, ds \\
&= 1 + \frac{5}{2}\, t^2 + \frac{5^2}{8}\, t^4 + \frac{5^3}{2^3 6}\, t^6.
\end{aligned}
$$

So we obtained $y_3(t) = 1 + \frac{5}{2} t^2 + \frac{5^2}{2^3}\, t^4 + \frac{5^3}{2^4 3}\, t^6$. We now rewrite this expression so we can get a power series expansion that can be written in terms of simple functions. The first step is to write the powers of t as t^n, for $n = 1, 2, 3,$

$$
y_3(t) = 1 + \frac{5}{2} (t^2)^1 + \frac{5^2}{2^3}\, (t^2)^2 + \frac{5^3}{2^4 3}\, (t^2)^3.
$$

Now we multiply by one each term to get the right facctorials, $n!$ on each term,

$$
y_3(t) = 1 + \frac{5}{2}\, \frac{(t^2)^1}{1!} + \frac{5^2}{2^2}\, \frac{(t^2)^2}{2!} + \frac{5^3}{2^3}\, \frac{(t^2)^3}{3!}.
$$

No we realize that the factor $5/2$ can be written together with the powers of t^2,

$$
y_3(t) = 1 + \frac{(\frac{5}{2} t^2)}{1!} + \frac{(\frac{5}{2}\, t^2)^2}{2!} + \frac{(\frac{5}{2}\, t^2)^3}{3!}.
$$

From this last expression is simple to guess the n-th approximation

$$
y_N(t) = 1 + \sum_{k=1}^{N} \frac{(\frac{5}{2} t^2)^k}{k!},
$$

which can be proven by induction. Therefore,

$$
y(t) = \lim_{N\to\infty} y_N(t) = 1 + \sum_{k=1}^{\infty} \frac{(\frac{5}{2} t^2)^k}{k!}.
$$

Recall now that the power series expansion for the exponential

$$
e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(at)^k}{k!}.
$$

so we get

$$
y(t) = 1 + (e^{\frac{5}{2} t^2} - 1) \quad \Rightarrow \quad y(t) = e^{\frac{5}{2} t^2}.
$$

$\triangleleft$

2.1. Second Order Linear Equations: General Properties

Section Objective(s):

- Second Order Linear Equations.
- Conservation of Mechanical Energy.
- Properties of Homogeneous Equations.

Remarks:

- We now study second order differential equations.
- The main example is Newton's law of motion.
- We have an existence result about solutions these equations without a formula for the solutions.
- We study ways to find properties of the solutions without solving the equations.
- One way is with the conservation of the energy
- We end this section studying properties of homogeneous equations.

2.1.1. Second Order Linear Equations.

Definition 1. A ***second order linear*** differential equation on y is

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t),$$

where a_1, a_0, b are given functions. The differential equation above:

(a) is ***homogeneous*** iff the source $b(t) = 0$ for all $t \in \mathbb{R}$;

(b) has ***constant coefficients*** iff a_1 and a_0 are constants;

(c) has ***variable coefficients*** iff either a_1 or a_0 is not constant.

Theorem 1. (IVP) If the a_1, a_0, b are continuous on (t_1, t_2) and $t_0 \in (t_1, t_2)$, then there is a unique y on (t_1, t_2) solution of the initial value problem

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_1.$$

Example 1. (Newton's Second Law of Motion): The main example of a second order linear equation is Newton's law of motion

$$ma = f \qquad a = y'' \quad \Rightarrow \quad m\, y'' = f.$$

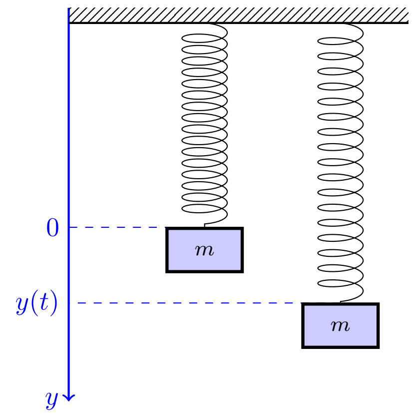
The moving particle is described by its position, velocity and acceleration.

- The function y is the position of a particle.
- The function $y' = v$ is the velocity of a particle.
- The function $a = y''$ is the acceleration of a particle.

The force may depend on time, position, and velocity. So, Newton's equation is the differential equation

$$m\, y'' = f(t, y, y').$$

Example 2. (Mass-Spring without Friction): Consider mass hanging at the bottom of a spring. Set y to be a vertical coordinate, with $y = 0$ at the equilibrium position of the mass-spring. Then, **Hooke's Law** states the force done by the spring on the mass is proportional to the stretching distance y and in the opposite to the stretching,



$$f = -ky, \qquad k > 0.$$

Newton's equation for this system, $m\,y'' = f$, is

$$m\,y'' = -k\,y \quad \Rightarrow \quad \boxed{m\,y'' + k\,y = 0}.$$

Example 3. (Mass-Spring with Friction): Consider mass hanging at the bottom of a spring describe in the example above. Suppose now that the whole system is oscillating inside a water bath. In this case appears an extra force, the friction between the oscillating mass and the water, given by

$$f_d = -d\,y', \qquad d > 0.$$

This friction, or damping force, opposes the movement. Then, Newton's equation, $m\,y'' = f$, in this case is

$$m\,y'' = -k\,y - d\,y' \quad \Rightarrow \quad \boxed{m\,y'' + d\,y' + k\,y = 0}.$$

2.1.2. Conservation of Mechanical Energy.

Theorem 2.1.1 (Conservation of the Energy)**.** *All solutions of the Mass-Spring System without friction*

$$m\,y'' + k\,y = 0,$$

satisfy that the quantity

$$E(t) = \frac{1}{2}m\,\big(v(t)\big)^2 + \frac{1}{2}k\,\big(y(t)\big)^2.$$

where $v = y'$ ____________, *is constant in time.*

Proof:

$$(m\,y'' + k\,y)\,y' = 0 \quad \Rightarrow \quad my''y' + kyy' = 0 \quad \Rightarrow \quad \frac{d}{dt}\Big(m\frac{(y')^2}{2} + k\frac{y^2}{2}\Big) = 0$$

$$E(t) = \frac{1}{2}\,m\,(y')^2 + \frac{1}{2}\,k\,y^2 \quad \Rightarrow \quad \frac{d}{dt}E(t) = 0.$$

$\lhd$

Example 4. (Conservation of the Energy): An object of mass $m = 1$ grams hanging at the bottom of a spring with a spring constant $k = 2$ grams per second square. Denote by y vertical coordinate, positive downwards, and $y = 0$ is the spring-mass resting position.

(1) Write the equation of motion for this object.

(2) Write the expression of the energy of this system.

(3) If the initial position of the object is $y(0) = 1$ and its initial velocity is $y(0) = 2$, find the maximum value of the object velocity, $v_{\max} > 0$ achieved during its motion.

Solution:

(1) The equation of motion is $y'' + 2\,y = 0$.

(2) The energy is obtained from

$$(y'' + 2y)\,y' = 0 \quad \Rightarrow \quad y''\,y' + y\,y' \quad \Rightarrow \quad \frac{d}{dt}\Big(\frac{1}{2}(y')^2 + y^2\Big) = 0$$

so the energy is $E(t) = \dfrac{1}{2}\,v^2 + y^2$, where $v = y'$.

(3) The energy is conserved: $E(t) = C$. And C is the initial energy $E(t) = E(0)$, so

$$\frac{1}{2}\,v^2(t) + y^2(t) = \frac{1}{2}v^2(0) + y^2(0).$$

But the initial conditions say that $y(0) = 1$, and $v(0) = 2$, so

$$\frac{1}{2}\,v^2(t) + y^2(t) = \frac{1}{2}\,4 + 1 \quad \Rightarrow \quad \frac{1}{2}\,v^2(t) + y^2(t) = 3.$$

From the expression above we see that the maximum speed $v_{\max}$ is achieved when $y(t) = 0$. At these times we get

$$\frac{1}{2}v_{\max}^2 = 3 \quad \Rightarrow \quad v_{\max} = \sqrt{6}.$$

$\triangleleft$

Example 1. (Extendability of Solutions): Find the maximum domain where the solution of the initial value problem below is certain to exist.

$$(t-1)\,y'' - 3t\,y' + \frac{4(t-1)}{(t-3)}\,y = t(t-1), \qquad y(2)=1, \qquad y'(2)=0.$$

Solution: The equation is $y'' - \dfrac{3t}{(t-1)}\,y' + \dfrac{4}{(t-3)}\,y = t$, so, the equation coefficients are defined on the domain

$$(-\infty,1)\cup(1,3)\cup(3,\infty) \quad \Rightarrow \quad \text{solution is defined on} \quad (-\infty,1) \quad \text{or} \quad (1,3) \quad \text{or} \quad (3,\infty)$$

The initial condition is at $t=2$, so the interval where the equation and the initial condition are defined is

$$D=(1,3).$$

$\triangleleft$

2.1.3. Properties of Homogeneous Equations.

Remark: We introduce the (operator) notation

$$y'' + a_1\, y' + a_0\, y = b(t) \quad \Leftrightarrow \quad L(y) = b(t) \quad \text{with} \quad L(y) = y'' + a_1\, y' + a_0\, y.$$

Theorem 2.1.5. (Superposition Property) If y_1, y_2 are solutions of the homogeneous equations $L(y_1) = 0$ and $L(y_2) = 0$, where $L(y) = y'' + a_1\, y' + a_0\, y$, then for every constants c_1, c_2 holds

$$L(c_1\, y_1 + c_2\, y_2) = 0.$$

Remark: This result is not true for nonhomogeneous equations.

Proof:

$$\begin{aligned} L(c_1\, y_1 + c_2\, y_2) &= (c_1\, y_1 + c_2\, y_2)'' + a_1\,(c_1\, y_1 + c_2\, y_2)' + a_0\,(c_1\, y_1 + c_2\, y_2) \\ &= c_1(y_1'' + a_1\, y_1' + a_0\, y_1) + c_2(y_2'' + a_1\, y_2' + a_0\, y_2) \\ &= c_1\, L(y_1) + c_2\, L(y_2) \\ &= 0. \end{aligned}$$

□

Theorem (General Solution). If y_1, y_2, with $y_1 \neq c\, y_2$ for any $c \in \mathbb{R}$, are solutions of $L(y_1) = 0$ and $L(y_2) = 0$, where $L(y) = y'' + a_1\, y' + a_0\, y$, then every solution y of $L(y) = 0$ can be written as

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t), \qquad c_1, c_2 \in \mathbb{R}.$$

Remark: Solutions y_1 and y_2 of $L(y) = 0$ with $y_1 \neq c\, y_2$ are called fundamental solutions.

Example 5. (Superposition Property): If y_1 is solution of

$$y'' + a_1\, y' + a_0\, y = 0, \tag{1}$$

and $y - 2$ is solution of

$$y'' + a_1\, y' + a_0\, y = \cos(2t), \tag{2}$$

then determine whether the following statements are True or False.

(1) $y_1 + y_2$ solves the homogeneous equation (1)

(2) $y_1 + y_2$ solves the non-homogeneous equation (2)

(3) $2\, y_1$ solves the homogeneous equation (1)

(4) $2\, y_2$ solves the non-homogeneous equation (2)

Solution:

(1) False

(2) True

(3) True

(4) False

◁

Example 6. (Fundamental Solutions): Show that $y_1 = e^t$ and $y_2 = e^{-2t}$ are fundamental solutions to the equation

$$y'' + y' - 2y = 0.$$

Solution: y_1, y_2 are l.i., so we only need to show that $L(y_1) = 0$ and $L(y_2) = 0$.

$$L(y_1) = y_1'' + y_1' - 2y_1 = e^t + e^t - 2e^t = (1 + 1 - 2)e^t = 0,$$

$$L(y_2) = y_2'' + y_2' - 2y_2 = 4\, e^{-2t} - 2\, e^{-2t} - 2e^{-2t} = (4 - 2 - 2)e^{-2t} = 0.$$

◁

Example 7.: Since $y_1 = 1$ is solution of

$$y'' + y' - 2y = -2.$$

find two more different solutions.

Solution:

$$y_2(t) = 1 + e^t, \qquad y_3(t) = 1 + e^{-2t}.$$

◁

2.2. Homogenous Constant Coefficients Equations

Section Objective(s):

Part 1:

- Review: General and Fundamental Solutions.
- Guessing Fundamental Solutions for 2×2 Systems.
- Solutions for 2×2 Systems.

Part 2:

- Review: Solutions for 2×2 Systems.
- The Complex Roots Case.
- Real Solutions for Complex Roots.

Remarks:

- Recall:

  Theorem (General Solution). If y_1, y_2, with $y_1 \neq c\, y_2$ for any $c \in \mathbb{R}$, are solutions of $L(y_1) = 0$ and $L(y_2) = 0$, where $L(y) = y'' + a_1\, y' + a_0\, y$, then every solution y of $L(y) = 0$ can be written as

  $$y(t) = c_1\, y_1(t) + c_2\, y_2(t), \qquad c_1, c_2 \in \mathbb{R}.$$

- Solutions y_1 and y_2 of $L(y) = 0$ with $y_1 \neq c\, y_2$ are called fundamental solutions.

- If we know 2 fundamental solutions, then we know all solutions of the homogeneous equation.

- For 2×2 system we guess the fundamental solutions.

2.2.1. Guessing Fundamental Solutions for 2×2 Systems.

Example 1. (Guessing Fundamental Solutions): Find all solutions to the equation

$$y'' + 5y' + 6y = 0.$$

Solution: Trial and Error:

- We try with simple functions: $y(t) = t^n$, $y(t) = e^{rt}$, $y(t) = \cos(at)$, etc.
- Power functions: $y = t^n$. We need $y' = n\,t^{n-1}$, and $y'' = n(n-1)\,t^{n-2}$.

$$n(n-1)\,t^{(n-2)} + 5n\,t^{(n-1)} + 6\,t^n = 0 \qquad \text{for all} \qquad t \in \mathbb{R} \quad \Rightarrow \quad \text{no solution.}$$

- Exponential functions: $y(t) = e^{rt}$. We need $y' = r\,e^{rt}$, and $y'' = r^2\,e^{rt}$.

$$r^2\,e^{rt} + 5r\,e^{rt} + 6\,e^{rt} = 0 \quad \Rightarrow \quad (r^2 + 5r + 6)\,e^{rt} = 0 \quad \Rightarrow \quad (r^2 + 5r + 6) = 0$$

Therefore we get two values: $r_1 = -2$ and $r_2 = -3$. So two **fundamental** solutions,

$$y_1 = e^{-2t}, \qquad y_2 = e^{-3t}.$$

The General solution Theorem Says the **general solution is**

$$y(t) = c\,e^{-2t} + c_2\,e^{-3t}.$$

$\triangleleft$

Definition 1. The ***characteristic polynomial*** and ***characteristic equation*** of the differential equation

$$y'' + a_1 y' + a_0 = 0 \quad a_1, a_0 \in \mathbb{R},$$

are, respectively,

$$p(r) = r^2 + a_1 r + a_0 \quad \text{and} \quad p(r) = 0.$$

Theorem 1. If $r_\pm$ are the roots of the characteristic polynomial of

$$y'' + a_1 y' + a_0 y = 0, \tag{1}$$

if c_+, c_- are arbitrary constants, then we have the following:

(a) If $\underline{r_+ \neq r_-}$, real or complex, then the general solution of Eq. (2) is

$$y_{\text{gen}}(t) = c_+ e^{r_+ t} + c_- e^{r_- t}.$$

(b) If $\underline{r_+ = r_- = r_0}$, real, then the general solution of Eq. (2) is

$$y_{\text{gen}}(t) = c_+ e^{r_0 t} + c_- t e^{r_0 t}.$$

Proof of Theorem 1:

Case (a): Since $r_+ \neq r_-$, then $e^{r_+ t} \neq c\, e^{r_- t}$, so we get $y_+ \neq c\, y_-$. Since $r_\pm$ are roots of the characteristic polynomial,

$$p(r_+) = 0, \qquad p(r_-) = 0,$$

then $y_\pm$ solve the differential equation. Indeed,

$$L(y_\pm) = (r_\pm^2 + a_1 r_\pm + a_0)\, e^{r_\pm t} = p(r_\pm)\, e^{r_\pm t} = 0.$$

Case (b): If $r_+ = r_- = r_0$, then we know that $y_1 = e^{r_0 t}$ is a solution, since

$$L(y_0) = (r_0^2 + a_1 r_0 + a_0)\, e^{r_0 t} = p(r_0)\, e^{r_0 t} = 0.$$

We now need to find a second solution y_2 not proportional to y_1. We use the **Reduction of Order Method:**

$$y_2(t) = v(t)\, y_1(t) \quad \Rightarrow \quad y_2(t) = v(t)\, e^{r_0 t},$$

and we put this expression in the differential equation (2),

$$\big(v'' + 2r_0 v' + v r_0^2\big)\, e^{r_0 t} + \big(v' + r_0 v\big)\, a_1 e^{r_0 t} + a_0 v\, e^{r_0 t} = 0.$$

We cancel the exponential out of the equation and we reorder terms,

$$v'' + (2r_0 + a_1)\, v' + (r_0^2 + a_1 r_0 + a_0)\, v = 0.$$

Recall that r_0 **is a root** of the characteristic polynomial

$$r_0^2 + a_1 r_0 + a_0 = 0,$$

Also recall that r_0 is the **only root**,

$$r_0 = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0} = -\frac{a_1}{2} \quad \Rightarrow \quad 2r_0 + a_1 = 0.$$

Therefore

$$v'' = 0 \quad \Rightarrow \quad v(t) = c_1 + c_2 t$$

and the second solution is

$$y_2(t) = (c_1 + c_2 t)\, y_1(t).$$

- Choosing $c_2 = 0$ is bad, y_2 is proportional to y_1.
- So $c_2 \neq 0$. We choose $c_2 = 1$, and we get

$$y_1(t) = e^{r_0 t}, \qquad y_2(t) = t\, e^{r_0 t}.$$

□

Example 2: Consider an object of mass $m = 1$ grams hanging from a spring with spring constant $k = 9$ grams per second square moving in a fluid with damping constant $d = 6$ grams per second. Find the position function of this object for arbitrary initial position and velocity.

Solution: The equation modeling the motion of the object is given by $y'' + 6y' + 9y = 0$.

The characteristic equation is

$$r^2 + 6r + 9 = 0 \quad \Rightarrow \quad r = \frac{1}{2}\big(-6 \pm \sqrt{36 - 36}\big) = -3, \quad \Rightarrow \quad r_+ = r_- = -3.$$

Therefore, the general solution of the equation above is

$$y_{\text{gen}}(t) = c_+ e^{-3t} + c_-\, t\, e^{-3t}.$$

$\lhd$

Example 3: Find the solution y of the initial value problem

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 5.$$

Solution: We find the roots of the characteristic polynomial

$$p(r) = r^2 - r - 2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big(1 \pm \sqrt{1+8}\big) = \frac{1 \pm 3}{2} \quad \Rightarrow \quad \begin{cases} r_+ = 2, \\ r_- = -1. \end{cases}$$

So the general solution is $y_{\text{gen}}(t) = c_+ e^{2t} + c_- e^{-t}$. The initial conditions fix c_+ and c_-, because

$$\left.\begin{array}{l} 1 = y(0) = c_+ + c_- \\ 5 = y'(0) = 2c_+ - c_- \end{array}\right\} \quad \Rightarrow \quad \begin{cases} c_+ = 2, \\ c_- = -1. \end{cases}$$

Therefore, the unique solution to the initial value problem is

$$y(t) = 2e^{2t} - e^{-t}.$$

$\lhd$

Section Objective(s):

Part 1:

- Review: General and Fundamental Solutions.
- Guessing Fundamental Solutions for 2×2 Systems.
- Solutions for 2×2 Systems.

Part 2:

- Review: Solutions for 2×2 Systems.
- The Complex Roots Case.
- Review of Complex Numbers.

Remarks:

- Recall the 2×2 case:

Theorem 1. If $r_\pm$ are the roots of the characteristic polynomial of

$$y'' + a_1 y' + a_0 y = 0, \tag{2}$$

and if c_+, c_- are arbitrary constants, then we have the following:

(a) If $r_+ \neq r_-$, real or complex, the general solution of Eq. (2) is

$$y_{\rm gen}(t) = c_+ \, e^{r_+ t} + c_- \, e^{r_- t}.$$

(b) If $r_+ = r_- = r_0$, real, the general solution of Eq. (2) is

$$y_{\rm gen}(t) = c_+ \, e^{r_0 t} + c_- \, t e^{r_0 t}.$$

- Equations with characteristic polynomial having complex roots have complex solutions.
- In some physical applications is important to have real solutions.
- Solutions of equations with complex roots describe dissipative phenomena.

2.2.2. The Complex Roots Case.

Example 4: Consider an object of mass $m = 1$ grams hanging from a spring with spring constant $k = 13$ grams per second square moving in a fluid with damping constant $d = 4$ grams per second. Find the position function of this object for arbitrary initial position and velocity.

Solution: The position y of the object must be solution of

$$y'' + 4y' + 13y = 0.$$

To find the solutions we first look for the roots of the characteristic polynomial,

$$r^2 + 4r + 13 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big(-4 \pm \sqrt{16 - 52}\big) \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big(-4 \pm \sqrt{36}\big),$$

so we obtain the roots

$$r_\pm = -2 \pm 3i.$$

Since the roots of the characteristic polynomial are different,

$$y_{\text{gen}}(t) = \tilde{c}_+ \, e^{(-2+3i)t} + \tilde{c}_- \, e^{(-2-3i)t}, \qquad \tilde{c}_+, \tilde{c}_- \in \mathbb{C}.$$

Unfortunately, it is not clear from the expression above how the object is going to move.

$\triangleleft$

2.2.3. Review of Complex Numbers.

Suppose that $a, b \in \mathbb{R}$. Then:

- Complex numbers have the form $z = a + ib$, where $i^2 = -1$.
- The complex conjugate of z is the number $\overline{z} = a - ib$.
- $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$ are the real and imaginary parts of z
- Hence: $\mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}$, $\mathrm{Im}(z) = \dfrac{z - \overline{z}}{2i}$.
- The **exponential of a complex number** is defined as

$$e^{a+ib} = \sum_{n=0}^{\infty} \frac{(a+ib)^n}{n!}.$$

  In particular, the following is true: $e^{a+ib} = e^a\, e^{ib}$.

- **Euler's formula:** $e^{ib} = \cos(b) + i \sin(b)$.
- Hence, a complex number of the form e^{a+ib} can be written as

$$e^{a+ib} = e^a \big(\cos(b) + i \sin(b)\big),$$

$$e^{a-ib} = e^a \big(\cos(b) - i \sin(b)\big).$$

- From e^{a+ib} and e^{a-ib} we get the real numbers

$$\frac{1}{2}\big(e^{a+ib} + e^{a-ib}\big) = e^a \cos(b),$$

$$\frac{1}{2i}\big(e^{a+ib} - e^{a-ib}\big) = e^a \sin(b).$$

Theorem 2. (Real Valued Fundamental Solutions) If the equation

$$y'' + a_1\, y' + a_0\, y = 0 \qquad \text{with} \qquad p(r) = r^2 + a_1 r + a_0$$

has coefficients such that $a_1{}^2 - 4a_0 < 0$ __________, then the roots of p are complex,

$$r_\pm = \alpha \pm i\beta \quad \text{with} \quad \alpha = -\frac{a_1}{2}, \quad \beta = \frac{1}{2}\sqrt{4a_0 - a_1^2},$$

and there are complex fundamental solutions of the differential equation,

$$\tilde{y}_+(t) = e^{(\alpha+i\beta)t}, \qquad \tilde{y}_-(t) = e^{(\alpha-i\beta)t},$$

while real valued fundamental solutions of the differential equation are

$$y_+(t) = e^{\alpha t}\cos(\beta t), \qquad y_-(t) = e^{\alpha t}\sin(\beta t).$$

Furthermore, the general solution of the differential equation can be written either as

$$y_{\text{gen}}(t) = \big(c_1 \cos(\beta t) + c_2 \sin(\beta t)\big)\, e^{\alpha t},$$

where c_1, c_2 are arbitrary constants, or as

$$y_{\text{gen}}(t) = A\, e^{\alpha t}\cos(\beta t - \phi)$$

where $A > 0$ is the amplitude __________ and $\phi \in [-\pi, \pi)$ is the phase shift __________.

Proof of Theorem 2: We start with the complex valued fundamental solutions

$$\tilde{y}_+(t) = e^{(\alpha+i\beta)t}, \qquad \tilde{y}_-(t) = e^{(\alpha-i\beta)t}.$$

We take the function $\tilde{y}_+$ and we use a property of complex exponentials,

$$\tilde{y}_+(t) = e^{(\alpha+i\beta)t} = e^{\alpha t}\, e^{i\beta t} = e^{\alpha t}\,\big(\cos(\beta t) + i\sin(\beta t)\big),$$

where we used Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Repeat this calculation for y_- we get,

$$\tilde{y}_+(t) = e^{\alpha t}\,\big(\cos(\beta t) + i\sin(\beta t)\big),$$

$$\tilde{y}_-(t) = e^{\alpha t}\,\big(\cos(\beta t) - i\sin(\beta t)\big).$$

The superposition property says that addition and differences of solutions to homogeneous equations are also solutions. So,

$$y_+(t) = \frac{1}{2}\big(\tilde{y}_+(t) + \tilde{y}_-(t)\big), \quad y_-(t) = \frac{1}{2i}\big(\tilde{y}_+(t) - \tilde{y}_-(t)\big),$$

are also solutions to the differential equation. But a straightforward computation gives

$$y_+(t) = e^{\alpha t}\cos(\beta t), \qquad y_-(t) = e^{\alpha t}\sin(\beta t).$$

Therefore, the genreal solution is

$$y_{\rm gen}(t) = \big(c_1 \cos(\omega_0 t) + c_2 \sin(\beta t)\big)\, e^{\alpha t}.$$

There is an equivalent way to express the general solution above given by

$$y_{\rm gen}(t) = A\, e^{\alpha t}\cos(\omega_0 t - \phi).$$

These two expressions for $y_{\rm gen}$ are equivalent because of the trigonometric identity

$$A\cos(\beta t - \phi) = A\cos(\beta t)\cos(\phi) + A\sin(\beta t)\sin(\phi),$$

which holds for all A and ϕ, and βt. Then, it is not difficult to see that

$$\left.\begin{aligned} c_1 &= A\cos(\phi), \\ c_2 &= A\sin(\phi). \end{aligned}\right\} \quad \Leftrightarrow \quad \left\{\begin{aligned} A &= \sqrt{c_1^2 + c_2^2}, \\ \tan(\phi) &= \frac{c_2}{c_1}. \end{aligned}\right.$$

This establishes the Theorem.

$\square$

Example 5. (Real Solutions - Mathematicians Notation): Describe the movement of the object in Example 4 above, which satisfies Newton's equation

$$y'' + 4y' + 13y = 0,$$

with initial position of 2 centimeters and initial velocity of 2 centimeters per second.

Solution: We already found the roots of the characteristic polynomial,

$$r^2 + 4r + 13 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big(-4 \pm \sqrt{16-52}\big) \quad \Rightarrow \quad r_\pm = -2 \pm 3i.$$

So the complex valued fundamental solutions are

$$\tilde{y}_+(t) = e^{(-2+3i)\,t}, \quad \tilde{y}_-(t) = e^{(-2-3i)\,t}.$$

We know that real valued fundamental solutions are given by

$$y_+(t) = e^{-2t}\cos(3t), \quad y_-(t) = e^{-2t}\sin(3t).$$

So the real valued general solution can be written as

$$y_{\text{gen}}(t) = \big(c_+ \cos(3t) + c_- \sin(3t)\big)\, e^{-2t}, \quad c_+,\ c_- \in \mathbb{R}.$$

We now use the initial conditions, $y(0) = 2$, and $y'(0) = 2$,

$$\left.\begin{aligned} 2 &= y(0) = c_+ \\ 2 &= y'(0) = 3c_- - 2c_+ \end{aligned}\right\} \quad \Rightarrow \quad c_+ = 2, \quad c_- = 2,$$

therefore the solution is

$$y(t) = \big(2\cos(3t) + 2\sin(3t)\big)\, e^{-2t}.$$

◁

Example 6. (Real Solution - Physicists Notation): Write the solution of the Example 5 above in terms of the amplitude A and phase shift ϕ.

Solution: To understand the movement of the object we write the solution in terms of amplitude and phase shift

$$y(t) = A\, e^{-2t} \cos(3t - \phi) \quad \Rightarrow \quad y'(t) = -2A\, e^{-2t} \cos(3t - \phi) - 3A\, e^{-2t} \sin(3t - \phi).$$

Let us use again the initial conditions $y(0) = 2$, and $y'(0) = 2$,

$$\left.\begin{aligned} 2 &= y(0) = A\cos(-\phi) \\ 2 &= y'(0) = -2A\cos(-\phi) - 3A\sin(-\phi) \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} 2 &= A\cos(\phi) \\ 2 &= -2A\cos(\phi) + 3A\sin(\phi) \end{aligned}\right.$$

Using the first equation in the second one we get

$$\left.\begin{aligned} 2 &= A\cos(\phi) \\ 2 &= -4 + 3A\sin(\phi) \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} 2 &= A\cos(\phi) \\ 2 &= A\sin(\phi) \end{aligned}\right.$$

From here it is not too difficult to see that

$$A = \sqrt{2^2 + 2^2} = 2\sqrt(2), \qquad \tan(\phi) = 1.$$

Since $\phi \in [-\pi, \pi)$, the equation $\tan(\phi) = 1$ has two solutions in that interval,

$$\phi_1 = \frac{\pi}{4}, \qquad \phi_2 = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}.$$

But the ϕ we need satisfies that $\cos(\phi) > 0$ and $\sin(\phi) > 0$, which means $\phi = \dfrac{\pi}{4}$, then

$$y(t) = 2\sqrt{2}\, e^{-2t} \cos\Big(3t - \frac{\pi}{4}\Big).$$

$\lhd$

Example 7. (Extra Problem): Find the movement of a 5kg mass attached to a spring with constant $k = 5\text{kg/secs}^2$ moving in a medium with damping constant $d = 5\text{kg/secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

Solution: The equation is $my'' + dy' + ky = 0$ with $m = 5$, $k = 5$, $d = 5$, that is,

$$y'' + y' + y = 0.$$

The roots of the characteristic polynomial are

$$r_\pm = \frac{1}{2}\big(-1 \pm \sqrt{1-4}\big) \quad \Rightarrow \quad r_\pm = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

We can write the solution in terms of an amplitude and a phase shift,

$$y(t) = A\, e^{-t/2} \cos\Big(\frac{\sqrt{3}}{2}\, t - \phi\Big).$$

We now use the initial conditions to find out the amplitude A and phase-shift ϕ. T

$$y'(t) = -\frac{1}{2}\, A\, e^{-t/2} \cos\Big(\frac{\sqrt{3}}{2}\, t - \phi\Big) - \frac{\sqrt{3}}{2}\, A\, e^{-t/2} \sin\Big(\frac{\sqrt{3}}{2}\, t - \phi\Big).$$

The initial conditions in the example imply,

$$\sqrt{3} = y(0) = A\cos(\phi), \qquad 0 = y'(0) = -\frac{1}{2}\, A\cos(\phi) + \frac{\sqrt{3}}{2}\, A\sin(\phi).$$

The second equation above allows us to compute the phase shift. Recall that $\phi \in [-\pi, \pi)$, and the condition that $\tan(\phi) = 1/\sqrt{3}$ has two solutions in that interval,

$$\tan(\phi) = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \phi_1 = \frac{\pi}{6}, \quad \text{or} \quad \phi_2 = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}.$$

If $\phi = -5\pi/6$, then $y(0) < 0$, which is not out case. Hence we **must choose** $\phi = \pi/6$.

$$\sqrt{3} = A\cos\Big(\frac{\pi}{6}\Big) = A\,\frac{\sqrt{3}}{2} \quad \Rightarrow \quad A = 2.$$

Therefore we obtain the solution

$$y(t) = 2\, e^{-t/2} \cos\Big(\frac{\sqrt{3}}{2}\, t - \frac{\pi}{6}\Big).$$

2.3. Nonhomogeneous Equations

Section Objective(s):

Part 1:

- The General Solution Theorem(NH).
- The Undetermined Coefficients Method.

Part 2:

- The Variation or Parameters Method.

Remarks:

- If y_1 and y_2 are solutions of the linear nonhomogeneous equation

$$y'' + a_1(t)\, y' + a_0(t)\, y = f,$$

is then $y_1 + y_2$ also a solution? And how about $5y_1$?

(1)

$$L(y_1) = f, \quad L(y_2) = f,$$

and

$$L(y_1 + y_2) = L(y_1) + L(y_2) = f + f = 2f.$$

(2)

$$L(5\,y - 1) = 5L(y_1) = 5\,f.$$

- Consider the following exercise:

(1) Guess a simple solution of $y'' + y = 7$.

$y_0 = 7.$

(2) Find fundamental solutions of $y'' + y = 0$.

$$r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i \quad \Rightarrow \quad y_1 = \cos(t), \quad y_2 = \sin(t)$$

(3) Now give 3 different solutions of $y'' + y = 7$.

$$y_1 = 7 + \sin(t), \quad y_2 = 7 + \cos(t), \quad y_3 = 7 + 2\sin(t) + 3\cos(t)$$

2.3.1. The General Solution Theorem.

Theorem 1. (General Solution (NH)) If y_1 and y_2 are fundamental solutions of

$$L(y_1) = 0, \qquad L(y_2) = 0,$$

where $L(y) = y'' + a_1\, y' + a_0\, y$, and y_p is one solution of $L(y_p) = f$, then all solutions of the nonhomogeneous equation $L(y) = f$ are

$$y = c_1\, y_1 + c_2\, y_2 + y_p, \qquad c_1\, c_2 \in \mathbb{R}.$$

Remark: The ***general solution*** of $L(y) = f$ is

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t) + y_p(t),$$

where y_p solves $L(y_p) = f$ and y_1, y_2 are fundamental solutions of $L(y) = 0$.

Proof of Theorem 1: Given any particular solution Let y_p, that is $L(y_p) = f$, any other solution y of the same equation $L(y) = f$ satisfies

$$L(y - y_p) = L(y) - L(y_p) = f - f = 0.$$

That is, $y - y_p$ is solution of the homogeneous equation. Therefore, this solution can be written as linear combinations of a pair of fundamental solutions, y_1, y_2 of the homogeneous equation,

$$y - y_p = c_1\, y_1 + c_2\, y_2.$$

Since for every y solution of $L(y) = f$ we can find constants c_1, c_2 such that the equation above holds true, we have found a formula for all solutions of the nonhomogeneous equation. This establishes the Theorem.

$\square$

2.3.2. The Undetermined Coefficients Method.

Example 1 (Guessing Solutions): If a_1, a_0 are arbitrary constants, guess a function y_p solution of

$$y'' + a_1\, y' + a_0\, y = 3\, e^{2t}$$

Solution:

- $f(t) = 3\, e^{2t} \quad \longrightarrow \quad y_p(t)$?
- Since $L(y) = y'' + a_1\, y' + a_0\, y$ has constant coefficients, we guess

$$y_p(t) = k\, e^{2t}.$$

Because

$$\begin{aligned} L(y_p) &= \big(k\, e^{2t}\big)'' + a_1\, \big(k\, e^{2t}\big) + a_0\, \big(k\, e^{2t}\big) \\ &= k\big(2^2 + a_1\, 2 + a_0\big)\, e^{2t} \\ &= k\, p(2)\, e^{2t}. \end{aligned}$$

It is a good idea to choose y_p proportional to e^{2t} because

$$\text{if} \quad L(y_p) = f, \quad \text{then}$$

$$k\, p(2)\, e^{2t} = 3\, e^{2t} \quad \Rightarrow \quad k\, p(2) = 3 \quad \Rightarrow \quad k = \frac{3}{p(2)}.$$

So we guessed right, and

$$y_p(t) = \frac{3}{p(2)}\, e^{2t}.$$

◁

Summary of the Undetermined Coefficients Method:

Problem Find y_p solution of $L(y_p) = f(t)$,

where $L(y) = y'' + a_1 y' + a_0 y$.

(1) **First Guess:** Given a simple $f(t)$, guess $y_{p_1}(t)$.

$f(t)$ (Source) (K, m, a, b, given.)	$y_p(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
Either t^m or $K_m t^m + \cdots + K_0$	$k_m t^m + \cdots + k_0$
$\cos(bt)$ and/or $\sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$

(2) **Possible Second Guess:** If y_{p_1} satisfies $L(y_{p_1}) = 0$,

then change the guess to $y_{p_2} = t\, y_{p_1}$.

(3) **Possible Third Guess:** If y_{p_2} satisfies $L(y_{p_2}) = 0$,

then change the guess to $y_{p_3} = t^2\, y_{p_2}$.

(4) **Find the Undetermined Coefficients:** From $L(y_p) = f$ get k,

where y_p is y_{p_1} or y_{p_2} or y_{p_3}.

Example 2. (First Guess Right): Find all solutions to the nonhomogeneous equation
$$y'' - 3y' - 4y = 3\,e^{2t}.$$

Solution: From the problem we get $L(y) = y'' - 3y' - 4y$ and $f(t) = 3e^{2t}$.

(1) Find fundamental solutions y_{+}, y_{-} to the homogeneous equation $L(y) = 0$. Since the homogeneous equation has constant coefficients we find the characteristic equation

$$r^2 - 3r - 4 = 0 \quad \Rightarrow \quad r_{+} = 4, \quad r_{-} = -1, \quad \Rightarrow \quad y_{+}(t) = e^{4t}, \quad y_{-} = (t) = e^{-t}.$$

(2) From the table: For $f(t) = 3\,e^{2t}$ guess $y_p(t) = k\,e^{2t}$. The constant k is the undetermined coefficient we must find.

(3) Since $y_p(t) = k\,e^{2t}$ is not solution of the homogeneous equation, we do not need to modify our guess. (Recall: $L(y) = 0$ iff exist constants c_{+}, c_{-} such that $y(t) = c_{+}\,e^{4t} + c_{-}\,e^{-t}$.)

(4) Introduce y_p into $L(y_p) = f$ and find k. So we do that,

$$(2^2 - 6 - 4)\,k\,e^{2t} = 3\,e^{2t} \quad \Rightarrow \quad -6k = 3 \quad \Rightarrow \quad k = -\frac{1}{2}.$$

We guessed that y_p must be proportional to the exponential e^{2t} in order to cancel out the exponentials in the equation above. We have obtained that

$$y_p(t) = -\frac{1}{2}\,e^{2t}.$$

The undetermined coefficients method gives us a way to compute a particular solution y_p of the nonhomogeneous equation. We now use the general solution theorem, Theorem 2.5.1, to write the general solution of the nonhomogeneous equation,

$$y_{\text{gen}}(t) = c_{+}\,e^{4t} + c_{-}\,e^{-t} - \frac{1}{2}\,e^{2t}.$$

$\triangleleft$

Example 3. (First Guess Wrong): Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\,e^{4t}.$$

Solution: If we write the equation as $L(y) = f$, with $f(t) = 3\,e^{4t}$, then the operator L is the same as in Example 2.5.1. So the solutions of the homogeneous equation $L(y) = 0$, are the same as in that example,

$$y_{+}(t) = e^{4t}, \qquad y_{-}(t) = e^{-t}.$$

The source function is $f(t) = 3\,e^{4t}$, so the Table 1 says that we need to guess $y_p(t) = k\,e^{4t}$. However, this function y_p is solution of the homogeneous equation, because

$$y_p = k\,y_{+} \quad \Rightarrow \quad Ly_p) = 0.$$

We have to change our guess, as indicated in the undetermined coefficients method, step (3)

$$y_p(t) = kt\,e^{4t}.$$

This new guess is not solution of the homogeneous equation. So we proceed to compute the constant k. We introduce the guess into $L(y_p) = f$,

$$y_p' = (1+4t)\,k\,e^{4t}, \qquad y_p'' = (8+16t)\,k\,e^{4t} \quad \Rightarrow \quad \big[8 - 3 + (16 - 12 - 4)t\big]\,k\,e^{4t} = 3\,e^{4t},$$

therefore, we get that

$$5k = 3 \quad \Rightarrow \quad k = \frac{3}{5} \quad \Rightarrow \quad y_p(t) = \frac{3}{5}\,t\,e^{4t}.$$

The general solution theorem for nonhomogneneous equations says that

$$y_{\text{gen}}(t) = c_{+}\,e^{4t} + c_{-}\,e^{-t} + \frac{3}{5}\,t\,e^{4t}.$$

$\triangleleft$

Example 4. (Extra Example: First Guess Right): Find all the solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: The equation is $L(y) = f$, with $f(t) = 2\sin(t)$ and L as in Example 1. So,

$$y_{+}(t) = e^{4t}, \qquad y_{-}(t) = e^{-t}, \quad \text{satisfy} \quad L(y_{+}) = 0 \qquad L(y_{-}) = 0.$$

Since $f(t) = 2\sin(t)$, we choose $y_p(t) = k_1\cos(t) + k_2\sin(t)$. This function y_p is not solution to the homogeneous equation. So we look for k_1, k_2 using the differential equation,

$$y_p' = -k_1\sin(t) + k_2\cos(t), \qquad y_p'' = -k_1\cos(t) - k_2\sin(t),$$

and then we obtain

$$[-k_1\cos(t) - k_2\sin(t)] - 3[-k_1\sin(t) + k_2\cos(t)] - 4[k_1\cos(t) + k_2\sin(t)] = 2\sin(t).$$

Reordering terms in the expression above we get

$$(-5k_1 - 3k_2)\cos(t) + (3k_1 - 5k_2)\sin(t) = 2\sin(t).$$

The last equation must hold for all $t \in \mathbb{R}$. In particular, it must hold for $t = \pi/2$ and for $t = 0$. At these two points we obtain, respectively,

$$\left.\begin{aligned} 3k_1 - 5k_2 &= 2, \\ -5k_1 - 3k_2 &= 0, \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} k_1 &= \frac{3}{17}, \\ k_2 &= -\frac{5}{17}. \end{aligned}\right.$$

So the particular solution to the nonhomogeneous equation is given by

$$y_p(t) = \frac{1}{17}\big[3\cos(t) - 5\sin(t)\big].$$

The general solution theorem for nonhomogeneous equations implies

$$y_{\text{gen}}(t) = c_{+}\, e^{4t} + c_{-}\, e^{-t} + \frac{1}{17}\big[3\cos(t) - 5\sin(t)\big].$$

$\lhd$

Section Objective(s):

Part 1:

- The General Solution Theorem(NH).
- The Undetermined Coefficients Method.

Part 2:

- The Variation or Parameters Method.

Remarks:

- Recall: The general solution of $L(y) = f$ is

$$y = c_1\, y_1 + c_2\, y_2 + y_p,$$

  where

$$L(y_1) = 0, \qquad L(y_2) = 0, \quad \text{and} \quad L(y_p) = f.$$

- The Undetermined Coefficients Method (UCM) is a way to guess y_p.
- The Variation of Parameters Method (VPM) gives a formula to y_p.
- VPM works on more general equations than the UCM.
- VPM works on $y'' + a_1(t)\, y' + a_0(t)\, y = f(t).$
- VPM usually takes longer to implement than the UCM.

2.3.3. The Variation of Parameters Method.

Theorem 1. (Variation of Parameters) A particular solution to the equation

$$L(y) = f,$$

with $L(y) = y'' + a_1(t)\, y' + a_0(t)\, y$ and a_1, a_0, f continuous functions, is given by

$$y_p = u_1 y_1 + u_2 y_2,$$

where y_1, y_2 are fundamental solutions of $L(y) = 0$ and u_1, u_2 are

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{12}(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{12}(t)}\, dt,$$

where W_{12} is the Wronskian of y_1 and y_2.

Remarks:

- The ***Wronskian*** of functions y_1 and y_2 is

$$W_{12}(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- If y_1 and y_2 are fundamental solutions of $y'' + a_1(t)\, y' + a_0(t)\, y = 0$,

  then $W_{12}(t) \neq 0$ for all t.

Proof of Theorem 1:

- The Reduction Order: $y_2 = v\, y_1$. Equation for v is simpler than for y_2.
- Here we have y_1 and y_2 and we look for y_p, so:

$$y_p = u_1\, y_1 + u_2\, y_2.$$

  We hope the equations for u_1, u_2 will be simpler than the equation for y_p.
- But we started with one unknown y_p, and now we have two unknowns u_1 and u_2.
- We are free to add one more equation to fix u_1, u_2.

- We choose

$$u_1'\,y_1 + u_2'\,y_2 = 0 \qquad \Big(\quad \Leftrightarrow \quad u_2 = \int -\frac{y_1'}{y_2'}\,u_1'\,dt \Big).$$

- We compute $L(y_p) = f$, we need $y_p = u_1\,y_1 + u_2\,y_2$, and y_p',

$$y_p' = u_1'\,y_1 + u_1\,y_1' + u_2'\,y_2 + u_2\,y_2' \quad \Rightarrow \quad y_p' = u_1\,y_1' + u_2\,y_2'.$$

(recall, $u_1'\,y_1 + u_2'\,y_2 = 0$) and we need y_p'',

$$y_p'' = u_1'\,y_1' + u_1\,y_1'' + u_2'\,y_2' + u_2\,y_2''.$$

- So the equation $L(y_p) = f$ is

$$(u_1'\,y_1' + u_1\,y_1'' + u_2'\,y_2' + u_2\,y_2'') + a_1(u_1\,y_1' + u_2\,y_2') + a_0(u_1\,y_1 + u_2\,y_2) = f$$

We reorder a few terms and we see that

$$u_1'\,y_1' + u_2'\,y_2' + u_1\,(y_1'' + a_1\,y_1' + a_0\,y_1) + u_2\,(y_2'' + a_1\,y_2' + a_0\,y_2) = f.$$

The functions y_1 and y_2 are solutions to the homogeneous equation,

$$y_1'' + a_1\,y_1' + a_0\,y_1 = 0, \qquad y_2'' + a_1\,y_2' + a_0\,y_2 = 0,$$

so u_1 and u_2 must be solution of a simpler equation that the one above, given by

$$u_1'\,y_1' + u_2'\,y_2' = f. \tag{1}$$

- So we end with the equations

$$u_1'\,y_1' + u_2'\,y_2' = f$$
$$u_1'\,y_1 + u_2'\,y_2 = 0.$$

- This is a 2×2 ***algebraic*** linear system for the unknowns u_1', u_2'.
- Algebraic linear systems are simple to solve

$$u_2' = -\frac{y_1}{y_2}\,u_1' \quad \Rightarrow \quad u_1'y_1' - \frac{y_1 y_2'}{y_2}\,u_1' = f \quad \Rightarrow \quad u_1'\Big(\frac{y_1'y_2 - y_1y_2'}{y_2}\Big) = f.$$

- Recall that the Wronskian of two functions is $W_{12} = y_1 y_2' - y_1' y_2$, we get

$$u_1' = -\frac{y_2 f}{W_{12}} \quad \Rightarrow \quad u_2' = \frac{y_1 f}{W_{12}}.$$

$\square$

Remark: The integration constants in $u_1,\ u_2$ can always be chosen zero.

Example 1: Find the general solution of the nonhomogeneous equation

$$y'' + 4y = -5\csc(2t).$$

Solution: We need fundamental solutions to the homogeneous problem.

$$r^2 + 4 = 0 \quad \Rightarrow \quad r_\pm = \pm 2i.$$

So, a pair of fundamental solutions is given by

$$y_1(t) = \cos(2t), \qquad y_2(t) = \sin(2t).$$

The Wronskian of these two functions is given by

$$W_{y_1 y_2}(t) = y_1 y_2' - y_2 y_1' = \cos(2t)\cdot 2\cdot\cos(2t) + \sin(2t)\cdot 2\cdot\sin(2t) = 2.$$

We are now ready to compute the functions u_1 and u_2.

$$u_1' = -\frac{y_2 f}{W_{12}}, \qquad u_2' = \frac{y_1 f}{W_{12}}.$$

So, the equation for u_1 is the following,

$$u_1' = \frac{5}{\sin(2t)}\,\frac{\sin(2t)}{2} = \frac{5}{2} \quad \Rightarrow \quad u_1 = \frac{5}{2}t,$$

$$u_2' = -\frac{5}{\sin(2t)}\,\frac{\cos(2t)}{2} \quad \Rightarrow \quad u_2 = -\frac{5}{2}\int \frac{\cos(2t)}{\sin(2t)}\,dt = -\frac{5}{4}\ln|\sin(2t)|,$$

where we have chosen the constants of integration to be zero. So,

$$y_p = \frac{5}{2}t\cos(2t) - \frac{5}{4}\ln|\sin(2t)|\cdot\sin(2t).$$

Then, $\quad y_{\text{gen}}(t) = c_+\cos(2t) + c_-\sin(2t) + \frac{5}{2}t\cos(2t) - \frac{5}{4}\ln|\sin(2t)|\cdot\sin(2t). \qquad c_+, c_- \in \mathbb{R}.$

$\triangleleft$

Example 2.: Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: We first rewrite the nonhomogeneous equation above in the form given in Theorem 2.5.4. In this case we must divide the whole equation by t^2,

$$y'' - \frac{2}{t^2}\, y = 3 - \frac{1}{t^2} \quad \Rightarrow \quad f(t) = 3 - \frac{1}{t^2}.$$

We now proceed to compute the Wronskian of the fundamental solutions y_1, y_2,

$$W_{12}(t) = (t^2)\Big(\frac{-1}{t^2}\Big) - (2t)\Big(\frac{1}{t}\Big) \quad \Rightarrow \quad W_{12}(t) = -3.$$

We now use the equation in Theorem 2.5.4 to obtain the functions u_1 and u_2,

$$\begin{aligned} u_1' &= -\frac{1}{t}\left(3 - \frac{1}{t^2}\right)\frac{1}{-3} & \qquad u_2' &= (t^2)\left(3 - \frac{1}{t^2}\right)\frac{1}{-3} \\ &= \frac{1}{t} - \frac{1}{3}\,t^{-3} \quad \Rightarrow \quad u_1 = \ln(t) + \frac{1}{6}t^{-2}, & &= -t^2 + \frac{1}{3} \quad \Rightarrow \quad u_2 = -\frac{1}{3}\,t^3 + \frac{1}{3}t. \end{aligned}$$

A particular solution to the nonhomogeneous equation above is $\tilde{y}_p = u_1 y_1 + u_2 y_2$, that is,

$$\begin{aligned} \tilde{y}_p &= \Big[\ln(t) + \frac{1}{6}t^{-2}\Big](t^2) + \frac{1}{3}(-t^3 + t)(t^{-1}) \\ &= t^2 \ln(t) + \frac{1}{6} - \frac{1}{3}\,t^2 + \frac{1}{3} \\ &= t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\,t^2 \\ &= t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\,y_1(t). \end{aligned}$$

However, a simpler expression for a solution of the nonhomogeneous equation above is

$$y_p = t^2 \ln(t) + \frac{1}{2}.$$

◁

Example 3. (Extra Example: Resonance): Consider a 1kg mass attached to a spring with a spring constant $k = 4\text{kg}/\text{sec}^2$. Assume that damping can be ignored and also assume there is an external force acting on the mass given by $f(t) = \sin(2t)$ acting on the body.

(1) Find an equation of the motion of the mass under general initial conditions.
(2) Describe the behavior of the amplitude of the oscillations as function of time.

Solution: The equation is

$$my'' + dy' + ky = f(t) \quad \Rightarrow \quad y'' + 4y = \sin(2t).$$

We use the undetermined coefficients. First we find the solutions to the homogeneous equation

$$y'' + 4y = 0 \quad \Rightarrow \quad r^2 + 4 = 0 \quad \Rightarrow \quad r_\pm = \pm 2i$$

So fundamental solutions are

$$y_1(t) = \cos(2t), \qquad y_2(t) = \sin(2t).$$

Therefore, the first gues for $y_p = k_1 \cos(2t) + k_2 \sin(2t)$ is wrong and we need to guess

$$y_p(t) = k_1 t \sin(2t) + k_2 t \cos(2t),$$

Put this into the nonhomogeneous equation and we get $k_1 = 0$ and $k_2 = -\dfrac{1}{4}$, so

$$y_p = -\frac{1}{4} t \cos(2t)$$

the general solution is given by

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) - \frac{1}{4} t \cos(2t).$$

This means that the amplitude **becomes unbounded** as time grows (about proportional to t). This is the resonance effect!

$\lhd$

2.4. Springs, Circuits, and Resonance

Section Objective(s):
- Springs and Circuits.
- Forced Oscillations:
  - Non-Resonant.
  - Resonant.

Remarks: Review of the superposition property.

- If y_1 and y_2 are solutions of the linear nonhomogeneous equation

$$y'' + a_1(t)\,y' + a_0(t)\,y = f,$$

is then $y_1 + y_2$ also a solution? And how about $5y_1$?

Question 1:

$$L(y_1) = f, \quad L(y_2) = f,$$

$$L(y_1 + y_2) = L(y_1) + L(y_2) = f + f = 2f.$$

Question 2:

$$L(5\,y_1) = 5\,L(y_1) = 5\,f.$$

- Guess a simple solution of $y'' + y = 7$.

$y_0 = 7$.

- Find fundamental solutions of $y'' + y = 0$.

$$r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i \quad \Rightarrow \quad y_1 = \cos(t), \quad y_2 = \sin(t)$$

- Now give 3 different solutions of $y'' + y = 7$.

$$y_1 = 7 + \sin(t), \quad y_2 = 7 + \cos(t), \quad y_3 = 7 + 2\sin(t) + 3\cos(t)$$

2.4.1. Springs and Circuits.

Spring Oscillations

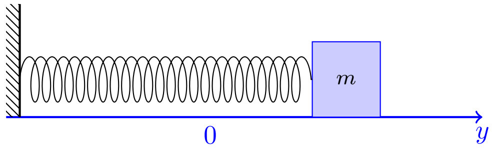


FIGURE 4. Mass-Spring system.

LC-Series Circuit

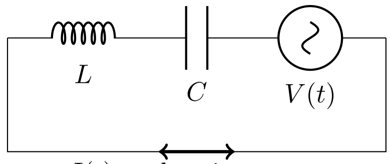


FIGURE 5. An LC circuit.

- L: inductance.
- C: capacitance.
- $V(t)$: voltage source.

Newton's equation:

$$my'' + ky = f(t),$$

with $f(t)$ an external force on the mass. If we set $m = 1$, $k = 25$, we get

$$\boxed{y'' + 25\,y = f(t)}.$$

Kirchhoff's equation:

$$LI'' + \frac{1}{C}\,I = V'(t).$$

If $L = 1$, $\frac{1}{C} = 25$, $f(t) = V'(t)$, we get

$$\boxed{y'' + 25\,y = f(t)}.$$

These Springs and Circuits are Mathematically Equivalent

Remark: Both systems are still mathematically equivalent in the presence of friction.

- Newton's equation with a friction coefficient $d > 0$ is

$$m\,y'' + d\,y' + k\,y = f(t), \qquad m > 0, \quad d > 0, \quad k > 0.$$

- Kirchhoff's equation with a resistance $R > 0$ is

$$L\,y'' + R\,y' + \frac{1}{C}\,y = V'(t), \qquad L > 0, \quad R > 0, \quad C > 0.$$

2.4.2. Forced Oscillations: Non-Resonant.

Example 1: (Non-Resonant): Solve the initial value problem

$$y'' + 25\,y = \cos(\nu t), \qquad \nu \neq 5, \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution:

Use the Undefined Coefficients Method. The gen. sol. of the homogeneous equation is

$$y_h(t) = c_1 \cos(5t) + c_2 \sin(5t).$$

the source is $f(t) = \cos(\nu t)$ with $\nu \neq 5$, so the correct guess for the particular solution is,

$$y_p(t) = k_1 \cos(\nu t) + k_2 \sin(\nu t).$$

We compute its second derivative,

$$y_p''(t) = -\nu^2 k_1 \cos(\nu t) - \nu^2 k_2 \sin(\nu t).$$

We substitute y and y'' in the non-homogeneous equation,

$$-\nu^2 k_1 \cos(\nu t) - \nu^2 k_2 \sin(\nu t) + 25k_1 \cos(\nu t) + 25k_2 \sin(\nu t) = \cos(\nu t).$$

$$\big(k_1(-\nu^2 + 25) - 1\big) \cos(\nu t) + k_2\,(-\nu^2 - 25) \sin(\nu t) = 0 \quad \Rightarrow \quad k_1 = \frac{1}{(25 - \nu^2)}, \quad k_2 = 0.$$

Therefore $y_p(t) = \dfrac{1}{(25 - \nu^2)} \cos(\nu t)$. The solution of the initial value problem is

$$y(t) = c_1 \cos(5t) + c_2 \sin(5t) + \frac{1}{(25 - \nu^2)} \cos(\nu t), \quad y(0) = 0, \quad y'(0) = 0.$$

The condition on $y(0) = 0$ implies

$$c_1 + \frac{1}{(25 - \nu^2)} = 0 \quad \Rightarrow \quad c_1 = -\frac{1}{(25 - \nu^2)}.$$

The condition $y(0) = 0$ implies $c_2 = 0$, so the solution of the initial value problem is

$$\boxed{y(t) = \frac{1}{(25 - \nu^2)} \big(\cos(\nu t) - \cos(5t)\big)}. \tag{1}$$

◁

2.4.3. Forced Oscillations: Non-Resonant.

Example 2: (Resonant): Solve the initial value problem

$$y'' + 25\,y = \cos(5t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution:

Use the Undefined Coefficients Method. The gen. sol. of the homogeneous equation is

$$y_h(t) = c_1 \cos(5t) + c_2 \sin(5t).$$

the source is $f(t) = \cos(5t)$, therefore the correct guess for the particular solution is,

$$y_p(t) = k_1 t \cos(5t) + k_2 t \sin(5t).$$

We compute its first derivative,

$$y_p'(t) = k_1 \cos(5t) + k_2 \sin(5t) - 5k_1 t \sin(5t) + 5k_2 t \cos(5t).$$

We compute the second derivative,

$$y_p''(t) = -10k_1 \sin(5t) + 10k_2 \cos(5t) - 25k_1 t \cos(2t) - 25k_2 t \sin(5t).$$

We substitute y and y'' in the nonhomogeneous equation,

$$-10k_1 \sin(5t) + 10k_2 \cos(5t) - 25k_1 t \cos(2t) - 25k_2 t \sin(5t) + 25k_1 t \cos(5t) + 25k_2 t \sin(5t) = \cos(5t).$$

$$-10k_1 \sin(5t) + 10k_2 \cos(5t) = \cos(5t) \quad \Rightarrow \quad k_1 = 0 \quad k_2 = \frac{1}{10}.$$

Therefore $y_p(t) = \dfrac{t}{10}\sin(5t)$. The solution of the initial value problem is

$$y(t) = c_1 \cos(5t) + c_2 \sin(5t) + \frac{1}{10} t \sin(5t), \quad y(0) = 0, \quad y'(0) = 0.$$

Since $y(0) = 0$ implies $c_1 = 0$, and $y'(0) = 0$ implies $c_2 = 0$, so $y(t) = y_p(t)$,

$$\boxed{y(t) = \frac{t}{10}\sin(5t)}.$$

$\lhd$

3.1. Introduction to the Laplace Transform

Section Objective(s):

- The Laplace Transform.
- Main Properties.
- Solving a Differential Equation.

Remarks:

- The Laplace Transform (LT) method introduces a new idea to solve differential equations.
- The idea is to use integration by parts.
- Because of that the LT changes derivatives into multiplications.
- So, LT changes differential equations into algebraic equations.

$$\mathcal{L}\begin{bmatrix}\text{differential} \\ \text{eq. for } y(t).\end{bmatrix} \xrightarrow{(1)} \begin{matrix}\text{Algebraic} \\ \text{eq. for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(2)} \begin{matrix}\text{Solve the} \\ \text{algebraic} \\ \text{eq. for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(3)} \begin{matrix}\text{Transform back} \\ \text{to obtain } y(t). \\ \text{(Use the table.)}\end{matrix}$$

- With the LT we can solve differential equations with general sources.
- Examples include:
  - Solve for the motion of objects hit by impulsive forces.
  - Solve for the current in an electric circuit having switches turn on and off.
- The Undetermined Coefficients Method is not powerful enough to solve differential equations with such general sources.

3.1.1. The Laplace Transform.

Definition 1. The ***Laplace transform*** of a function f on $D_f = [0, \infty)$ is

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

defined for all $s \in D_F \subset \mathbb{R}$ where the integral converges.

Remarks:

(a) Transformation notations for the Laplace transform: $\mathcal{L}[f] = F$.

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} \,(f(t))\, dt.$$

(b) Recall the definition of improper integrals:

$$\int_0^\infty g(t)\, dt = \lim_{N\to\infty} \int_0^N g(t)\, dt.$$

Example 1. (Computing a LT): Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.

Solution: We start with the definition of the Laplace transform,

$$\begin{aligned}
\mathcal{L}[e^{at}] &= \int_0^\infty e^{-st}(e^{at})\, dt \\
&= \int_0^\infty e^{(a-s)t}\, dt \quad \overset{(s=a)}{\Rightarrow} \quad \int_0^\infty 1\, dt = \infty, \\
&= \lim_{N\to\infty} \int_0^N e^{(a-s)t}\, dt, \qquad s \neq a, \\
&= \lim_{N\to\infty} \Big[\frac{1}{(a-s)}\, e^{(a-s)t} \Big|_0^N \Big] \\
&= \frac{1}{(a-s)} \Big(\lim_{N\to\infty} e^{(a-s)N} - 1 \Big).
\end{aligned}$$

Now we have two remaining cases:

$$a - s > 0 \quad \Rightarrow \quad \lim_{N\to\infty} e^{(a-s)N} = \infty \quad \text{and} \quad a - s < 0 \quad \Rightarrow \quad \Rightarrow \quad \lim_{N\to\infty} e^{(a-s)N} = 0,$$

so the integral converges only for $s > a$ and the Laplace transform is given by

$$\mathcal{L}[e^{at}] = \frac{1}{(s-a)}, \qquad s > a.$$

$\triangleleft$

3.1.2. Main Properties.

Linearity

Theorem 1. (Linearity) If $\mathcal{L}[f]$ and $\mathcal{L}[g]$ exist, then for all $a,\, b \in \mathbb{R}$ holds

$$\mathcal{L}[af + bg] = a\,\mathcal{L}[f] + b\,\mathcal{L}[g].$$

Proof of Theorem 1:

Let f and g be two functions, such that $\mathcal{L}[f]$ and $\mathcal{L}[g]$ are defined and let $c_1, c_2 \in \mathbb{R}$. Then,

$$\begin{aligned}
\mathcal{L}[c_1 f + c_2 g](s) &= \int_0^\infty e^{-st}(c_1 f(t) + c_2 g(t))dt \\
&= c_1 \int_0^\infty e^{-st} f(t)dt + c_2 \int_0^\infty e^{-st} g(t)dt \\
&= c_1 \mathcal{L}[f](s) + c_2 \mathcal{L}[g](s).
\end{aligned}$$

□

Remarks:

(a) A particular case of linearity is $\mathcal{L}[c\, f(t)] = c\,\mathcal{L}[f(t)$.

(b) In the case that c is not constant we have $\mathcal{L}[c(t)\, f(t)] \neq c(t)\,\mathcal{L}[f(t)]$.

(c) Also, when c is not a constant $\mathcal{L}[c(t)\, f(t)] \neq \mathcal{L}[c(t)]\,\mathcal{L}[f(t)]$.

Derivatives into Multiplication

Theorem 2. (Derivative into Multiplication) If both f and f' are continuous and $|f(t)| \leqslant k\, e^{at}$, with $k, a > 0$, all conditions on $[0, \infty)$, then $\mathcal{L}[f']$ exists for $s > a$ and

$$\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0), \qquad s > a.$$

Proof of Theorem 2:

We compute

$$\mathcal{L}[f'] = \lim_{N\to\infty} \int_0^N e^{-st}\, f'(t)\, dt.$$

But

$$\int_0^N e^{-st} f'(t)\, dt = \Big[\Big(e^{-st} f(t)\Big)\Big|_0^N - \int_0^N (-s) e^{-st} f(t)\, dt\Big]$$
$$= e^{-sN} f(N) - f(0) + s \int_0^N e^{-st} f(t)\, dt.$$

Since f is bounded by an exponential, this means

$$\lim_{n\to\infty} e^{-sN} f(N) = 0.$$

We now compute the limit of this expression above as $N \to \infty$.

$$\mathcal{L}[f'] = -f(0) + s \lim_{N\to\infty} \int_0^N e^{-st} f(t)\, dt.$$

Therefore,

$$\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0), \qquad s > a.$$

This establishes the Theorem. $\square$

Exercise: Use the formula above to compute the LT of second (and higher) derivatives,

$$\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\, f(0) - f'(0).$$

$$\mathcal{L}[f''] = \mathcal{L}[(f')'] = s\mathcal{L}[f'] - f'(0) = s\big(s\mathcal{L}[f] - f(0)\big) - f'(0).$$

Laplace Transform Table: We collect the LT of simple functions.

$f(t)$	$F(s) = \mathcal{L}[f(t)]$	D_F
$f(t) = 1$	$F(s) = \dfrac{1}{s}$	$s > 0$
$f(t) = e^{at}$	$F(s) = \dfrac{1}{(s-a)}$	$s > a$
$f(t) = t^n$	$F(s) = \dfrac{n!}{s^{(n+1)}}$	$s > 0$
$f(t) = \sin(at)$	$F(s) = \dfrac{a}{s^2 + a^2}$	$s > 0$
$f(t) = \cos(at)$	$F(s) = \dfrac{s}{s^2 + a^2}$	$s > 0$
$f(t) = \sinh(at)$	$F(s) = \dfrac{a}{s^2 - a^2}$	$s > \|a\|$
$f(t) = \cosh(at)$	$F(s) = \dfrac{s}{s^2 - a^2}$	$s > \|a\|$
$f(t) = t^n e^{at}$	$F(s) = \dfrac{n!}{(s-a)^{(n+1)}}$	$s > a$
$f(t) = e^{at}\sin(bt)$	$F(s) = \dfrac{b}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\cos(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\sinh(bt)$	$F(s) = \dfrac{b}{(s-a)^2 - b^2}$	$s - a > \|b\|$
$f(t) = e^{at}\cosh(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 - b^2}$	$s - a > \|b\|$

3.1.3. Solving a Differential Equation.

Example 2. (Solving an IVP): Use the Laplace transform to find y, solution of

$$y' = -5\,y, \qquad y(0) = 2.$$

Solution: **Remark:** We know that the solution is $y(t) = 2\,e^{-5t}$.

We write the equation as

$$y' + 5y = 0.$$

Taking the Laplace Transform of the ODE yields

$$\mathcal{L}[y' + 5y] = \mathcal{L}[0] \quad \Rightarrow \quad \mathcal{L}[y'] + 5\,\mathcal{L}[y] = 0.$$

But the LT changes derivatives into multiplications,

$$(s\,\mathcal{L}[y] - 2) + 5\,\mathcal{L}[y] = 0.$$

Using the notation $Y(s) = \mathcal{L}[y]$, we get

$$s\,Y(s) - 2 + 5\,Y(s) = 0 \quad \Rightarrow \quad (s+5)\,Y(s) = 2 \quad \Rightarrow \quad Y(s) = \frac{2}{(s+5)}.$$

From the LT table we see that $\mathcal{L}[e^{at}] = \dfrac{1}{(s-a)}$, so for $a = -5$ we get

$$\mathcal{L}[y] = 2\mathcal{L}[e^{-5t}] \quad \Rightarrow \quad \mathcal{L}[y] = \mathcal{L}[2\,e^{-5t}].$$

We then conclude that

$$y(t) = 2\,e^{-5t}.$$

$\lhd$

Example 3. (Extra Example): Find the solution of the IVP

$$y'' - 4\,y' + 13\,y = 0, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Solution:

Taking the LT,

$$\mathcal{L}[y'' - 4y' + 13y] = 0 \quad \Rightarrow \quad \mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 13\,\mathcal{L}[y] = 0.$$

We change derivatives into multiplication,

$$\big(s^2\,Y(s) - s\,y(0) - y'(0)\big) - 4\big(s\,Y(s) - y(0)\big) + 13\,Y(s) = 0$$

$$(s^2 - 4s + 13)\,Y(s) = s\,y(0) + y'(0) - 4y(0) \quad \Rightarrow \quad (s^2 - 4s + 13)\,Y(s) = 1$$

Therefore we get

$$Y(s) = \frac{1}{s^2 - 4s + 13}$$

But

$$s^2 - 4s + 13 = s^2 - 2(2)s + 4 - 4 + 13 = (s-2)^2 + 9$$

Therefore,

$$\mathcal{L}[y] = \frac{1}{(s-2)^2 + 3^2} = \frac{1}{3}\Big(\frac{3}{(s-2)^2 + 3^2}\Big)$$

In the LT Table we have $\mathcal{L}[e^{at}\sin(bt)] = \dfrac{b}{(s-a)^2 + b^2}$. Therefore,

$$\mathcal{L}[y] = \frac{1}{3}\mathcal{L}[e^{2t}\sin(3t)].$$

We then conclude

$$y(t) = \frac{1}{3}\,e^{2t}\sin(3t).$$

$\triangleleft$

Example 3. (Compute Another LT): Compute $\mathcal{L}[\sin(at)]$, where $a \in \mathbb{R}$.

Solution: In this case we need to compute

$$\begin{aligned}\mathcal{L}[\sin(at)] &= \int_0^\infty e^{-st}\sin(at)\,dt\\ &= \lim_{N\to\infty}\int_0^N e^{-st}\sin(at)\,dt\end{aligned}$$

The definite integral above can be computed integrating by parts twice,

$$\int_0^N e^{-st}\sin(at)\,dt = -\frac{1}{s}\big[e^{-st}\sin(at)\big]\Big|_0^N - \frac{a}{s^2}\big[e^{-st}\cos(at)\big]\Big|_0^N - \frac{a^2}{s^2}\int_0^N e^{-st}\sin(at)\,dt,$$

which implies that

$$\Big(1+\frac{a^2}{s^2}\Big)\int_0^N e^{-st}\sin(at)\,dt = -\frac{1}{s}\big[e^{-st}\sin(at)\big]\Big|_0^N - \frac{a}{s^2}\big[e^{-st}\cos(at)\big]\Big|_0^N.$$

then we get

$$\int_0^N e^{-st}\sin(at)\,dt = \frac{s^2}{(s^2+a^2)}\Big[-\frac{1}{s}\big[e^{-st}\sin(at)\big]\Big|_0^N - \frac{a}{s^2}\big[e^{-st}\cos(at)\big]\Big|_0^N\Big].$$

and finally we get

$$\int_0^N e^{-st}\sin(at)\,dt = \frac{s^2}{(s^2+a^2)}\Big[-\frac{1}{s}\big[e^{-sN}\sin(aN)-0\big] - \frac{a}{s^2}\big[e^{-sN}\cos(aN)-1\big]\Big].$$

One can check that the limit $N\to\infty$ on the right hand side above does not exist for $s \leqslant 0$, so $\mathcal{L}[\sin(at)]$ does not exist for $s \leqslant 0$. In the case $s > 0$ it is not difficult to see that

$$\int_0^\infty e^{-st}\sin(at)\,dt = \Big(\frac{s^2}{s^2+a^2}\Big)\Big[\frac{1}{s}\,(0-0) - \frac{a}{s^2}\,(0-1)\Big]$$

so we obtain the final result

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}, \qquad s > 0.$$

$\lhd$

3.2. The Initial Value Problem

Section Objective(s):

- Homogeneous IVP.
- Non-Homogeneous IVP.
- Higher Order IVP.

When can we apply the Laplace Transform Method?

(a) The ODEs have to be linear with constant coefficients.

(b) The sources can be discontinuous or Dirac's deltas.

The big picture approach in using the LT to solve ODEs:

$$\mathcal{L}\begin{bmatrix}\text{differential eq.}\\ \text{for } y(t).\end{bmatrix} \xrightarrow{(1)} \begin{matrix}\text{Algebraic eq.}\\ \text{for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(2)} \begin{matrix}\text{Solve the}\\ \text{algebraic eq.}\\ \text{for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(3)} \begin{matrix}\text{Transform back}\\ \text{to obtain } y(t).\\ \text{(Use the table.)}\end{matrix}$$

The One-to-One Property:

Theorem 1. (Injectivity of the Laplace Transform) If f, g are continuous on $[0, \infty)$ and bounded by an exponential, then

$$\mathcal{L}[f] = \mathcal{L}[g] \quad \Rightarrow \quad f = g.$$

Remark: We use one-to-one property when we solve differential equations.

- If we LT a differential equation for y and solve for $\mathcal{L}[y]$, we get

$$\mathcal{L}[y(t)] = H(s),$$

- We find $h(t)$ such that $\mathcal{L}[h] = H$, so we get

$$\mathcal{L}[y(t)] = \mathcal{L}[h(t)] \quad \Rightarrow \quad y(t) = h(t).$$

3.2.1. Homogeneous IVP.

Example 1: Use the Laplace transform to find the solution y to the initial value problem

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: First, compute the Laplace transform of the differential equation,

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0] = 0.$$

The Laplace transform is a linear operation,

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\,\mathcal{L}[y] = 0.$$

We know that the LT relates derivatives to multiplications,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - \Big[s\,\mathcal{L}[y] - y(0)\Big] - 2\,\mathcal{L}[y] = 0,$$

which is equivalent to the equation

$$(s^2 - s - 2)\,\mathcal{L}[y] = (s-1)\,y(0) + y'(0).$$

The differential equation for y is now an algebraic equation for $\mathcal{L}[y]$. The initial condition,

$$(s^2 - s - 2)\,\mathcal{L}[y] = (s-1).$$

Solve for the unknown $\mathcal{L}[y]$ as follows,

$$\mathcal{L}[y] = \frac{(s-1)}{(s^2 - s - 2)}.$$

The function on the right-hand side above ***does not*** appear in our LT Table, so we use ***partial fractions*** to simplify it. First find the roots of the polynomial in the denominator,

$$s^2 - s - 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\big[1 \pm \sqrt{1+8}\big] \quad \Rightarrow \quad \begin{cases} s_+ = 2, \\ s_- = -1, \end{cases}$$

that is, the polynomial has two real roots. In this case we factorize the denominator,

$$\mathcal{L}[y] = \frac{(s-1)}{(s-2)(s+1)}.$$

The ***partial fraction*** decomposition of the right-hand side in the equation above is the following: Find constants a and b such that

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}.$$

A simple calculation shows

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a\,(s+1) + b\,(s-2)}{(s-2)(s+1)}.$$

Hence constants a and b must be solutions of the equations

$$(s-1) = a\,(s+1) + b\,(s-2)$$

Evaluate the equation above at $s=2$ and $s=-1$. We get

$$\text{If} \quad s=2 \quad \Rightarrow \quad (2-1) = a\,(2+1) + 0 \quad \Rightarrow \quad a = \frac{1}{3},$$
$$\text{If} \quad s=-1 \quad \Rightarrow \quad (-1-1) = 0 + b\,(-1-2) \quad \Rightarrow \quad a = \frac{2}{3}.$$

Hence,

$$\mathcal{L}[y] = \frac{1}{3}\,\frac{1}{(s-2)} + \frac{2}{3}\,\frac{1}{(s+1)}.$$

Using the Laplace transform table given in the previous class, we know that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{1}{s-2} = \mathcal{L}[e^{2t}], \qquad \frac{1}{s+1} = \mathcal{L}[e^{-t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \frac{1}{3}\,\mathcal{L}[e^{2t}] + \frac{2}{3}\,\mathcal{L}[e^{-t}] = \mathcal{L}\Big[\frac{1}{3}\big(e^{2t} + 2e^{-t}\big)\Big]$$

We conclude that

$$y(t) = \frac{1}{3}\big(e^{2t} + 2e^{-t}\big).$$

$\triangleleft$

3.2.2. Non-Homogeneous IVP.

Example 2: Use the Laplace transform to find the solution y to the initial value problem

$$y'' - 4y' + 4y = 3\,e^t, \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: First, compute the Laplace transform of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3\,e^t] = 3\Big(\frac{1}{s-1}\Big).$$

The Laplace transform is a linear operation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \frac{3}{s-1}.$$

The Laplace transform relates derivatives with multiplication,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = \frac{3}{s-1},$$

But the initial conditions are $y(0) = 0$ and $y'(0) = 0$, so

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = \frac{3}{s-1}.$$

Solve the algebraic equation for $\mathcal{L}[y]$,

$$\mathcal{L}[y] = \frac{3}{(s-1)(s^2 - 4s + 4)}.$$

We use ***partial fractions*** to simplify the right-hand side above. We start finding the roots of the polynomial in the denominator,

$$s^2 - 4s + 4 = 0 \quad\Rightarrow\quad s_\pm = \frac{1}{2}\big[4 \pm \sqrt{16-16}\big] \quad\Rightarrow\quad s_+ = s_- = 2.$$

that is, the polynomial has a single real root, so $\mathcal{L}[y]$ can be written as

$$\mathcal{L}[y] = \frac{3}{(s-1)(s-2)^2}.$$

The partial fraction decomposition of the righthand side above is

$$\frac{3}{(s-1)(s-2)^2} = \frac{a}{(s-1)} + \frac{b\,s + c}{(s-2)^2} = \frac{a\,(s-2)^2 + (b\,s+c)(s-1)}{(s-1)(s-2)^2}$$

From the far right and left expressions above we get

$$3 = a\,(s-2)^2 + (b\,s+c)(s-1) = a\,(s^2-4s+4) + b\,s^2 - b\,s + c\,s - c$$

Expanding all terms above, and reordering terms, we get

$$(a+b)\,s^2 + (-4a-b+c)\,s + (4a-c-3) = 0.$$

Since this polynomial in s vanishes for all $s \in \mathbb{R}$, we get that

$$\left.\begin{aligned} a+b&=0,\\ -4a-b+c&=0,\\ 4a-c-3&=0. \end{aligned}\right\} \quad\Rightarrow\quad \left\{\begin{aligned} a&=3\\ b&=-3\\ c&=9. \end{aligned}\right.$$

So we get

$$\mathcal{L}[y] = \frac{3}{(s-1)(s-2)^2} = \frac{3}{s-1} + \frac{-3s+9}{(s-2)^2}$$

One last trick is needed on the last term above,

$$\frac{-3s+9}{(s-2)^2} = \frac{-3(s-2+2)+9}{(s-2)^2} = \frac{-3(s-2)}{(s-2)^2} + \frac{-6+9}{(s-2)^2} = -\frac{3}{(s-2)} + \frac{3}{(s-2)^2}.$$

So we finally get

$$\mathcal{L}[y] = \frac{3}{s-1} - \frac{3}{(s-2)} + \frac{3}{(s-2)^2}.$$

From our Laplace transforms Table we know that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad\Rightarrow\quad \frac{1}{s-2} = \mathcal{L}[e^{2t}],$$

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2} \quad\Rightarrow\quad \frac{1}{(s-2)^2} = \mathcal{L}[te^{2t}].$$

So we arrive at the formula

$$\mathcal{L}[y] = 3\,\mathcal{L}[e^t] - 3\,\mathcal{L}[e^{2t}] + 3\,\mathcal{L}[t\,e^{2t}] = \mathcal{L}\big[3\,(e^t - e^{2t} + t\,e^{2t})\big]$$

So we conclude that $\quad y(t) = 3\,(e^t - e^{2t} + t\,e^{2t}).$ $\lhd$

3.2.3. Higher Order IVP.

Example 3: Use the Laplace transform to find the solution y to the initial value problem

$$y^{(4)} - 4y = 0, \qquad \begin{array}{ll} y(0) = 1, & y'(0) = 0, \\ y''(0) = -2, & y'''(0) = 0. \end{array}$$

Solution: Compute the Laplace transform of the differential equation,

$$\mathcal{L}[y^{(4)} - 4y] = \mathcal{L}[0] = 0.$$

The Laplace transform is a linear operation,

$$\mathcal{L}[y^{(4)}] - 4\,\mathcal{L}[y] = 0,$$

and the Laplace transform relates derivatives with multiplications,

$$\Big[s^4\,\mathcal{L}[y] - s^3\,y(0) - s^2\,y'(0) - s\,y''(0) - y'''(0)\Big] - 4\,\mathcal{L}[y] = 0.$$

From the initial conditions we get

$$\Big[s^4\,\mathcal{L}[y] - s^3 - 0 + 2s - 0\Big] - 4\,\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s^4 - 4)\mathcal{L}[y] = s^3 - 2s \quad \Rightarrow \quad \mathcal{L}[y] = \frac{(s^3 - 2s)}{(s^4 - 4)}.$$

In this case we are lucky, because

$$\mathcal{L}[y] = \frac{s(s^2 - 2)}{(s^2 - 2)(s^2 + 2)} = \frac{s}{(s^2 + 2)}.$$

Since

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2},$$

we get that

$$\mathcal{L}[y] = \mathcal{L}[\cos(\sqrt{2}t)] \quad \Rightarrow \quad y(t) = \cos(\sqrt{2}t).$$

$\triangleleft$

3.3. Discontinuous Sources

Section Objective(s):

- Overview: Step Functions.
- Laplace Transform of Steps.
- Translation Properties of the LT.

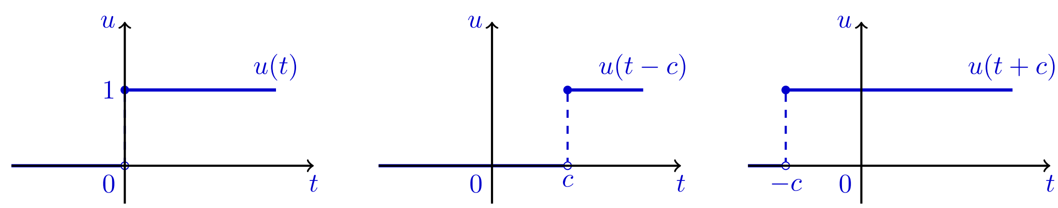
3.3.1. Overview: Step Functions.

Definition 1. The ***step function*** at $t = 0$ is

$$u(t) = \begin{cases} \underline{0\qquad\qquad} & t < 0, \\ \underline{1\qquad\qquad} & t \geqslant 0. \end{cases}$$

Example 1: Graph the step u, $u_c(t) = u(t-c)$, and $u_{-c}(t) = u(t+c)$, for $c > 0$.

Solution:



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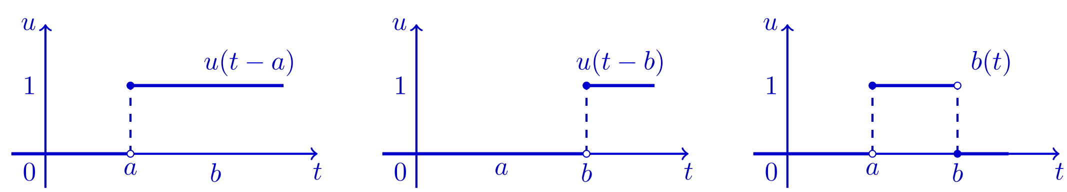
Example 2: Graph the bump function $b(t) = u(t-a) - u(t-b)$, for $a < b$.

Solution: The bump function b is nonzero only on a finite interval $[a, b]$, because

$$b(t) = u(t-a) - u(t-b) \quad \Leftrightarrow \quad b(t) = \begin{cases} 0 & t < a, \\ 1 & a \leqslant t < b \\ 0 & t \geqslant b. \end{cases}$$



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3.3.2. Translation Identities.

Theorem 2. (Translation Identities) If $\mathcal{L}[f(t)](s)$ exists for $s > a$, then

$$\mathcal{L}[u(t-c)f(t-c)] = \underline{e^{-cs}\,\mathcal{L}[f(t)]\qquad\qquad}, \qquad s > a, \qquad c \geqslant 0 \qquad (1)$$

$$\mathcal{L}[e^{ct}f(t)] = \underline{\mathcal{L}[f(t)](s-c)\qquad\qquad}, \qquad s > a + c, \qquad c \in \mathbb{R}. \qquad (2)$$

Example 3: Take $f(t) = \cos(2t)$ and write the equations given the Theorem above.

Solution:

$$\mathcal{L}[\cos(2t)] = \frac{s}{s^2 + 2^2}$$

$$\mathcal{L}[u(t-c)\,\cos(2(t-c))] = e^{-cs}\,\mathcal{L}[\cos(2t)] \quad \Rightarrow \quad \mathcal{L}[u(t-c)\,\cos(2(t-c))] = e^{-cs}\,\frac{s}{s^2+2^2}.$$

$$\mathcal{L}[e^{ct}\cos(2t)] = \mathcal{L}[\cos(2t)](s-c) \quad \Rightarrow \quad \mathcal{L}[e^{ct}\cos(2t)] = \frac{(s-c)}{(s-c)^2+2^2}.$$

◁

Example 4: Take $f(t) = 1$ and write the equations given the Theorem above.

Solution:

$$\mathcal{L}[1] = \frac{1}{s}$$

$$\mathcal{L}[u(t-c)] = e^{-cs}\,\mathcal{L}[1] \quad \Rightarrow \quad \boxed{\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}}.$$

$$\mathcal{L}[e^{ct}] = \mathcal{L}[1](s-c) \quad \Rightarrow \quad \mathcal{L}[e^{ct}] = \frac{1}{(s-c)}.$$

◁

Example 5: Find the function f such that $\mathcal{L}[f(t)] = \dfrac{e^{-4s}}{s^2+5}$.

Solution: Notice that

$$\mathcal{L}[f(t)] = e^{-4s}\,\frac{1}{s^2+5} \quad \Rightarrow \quad \mathcal{L}[f(t)] = \frac{1}{\sqrt{5}}\,e^{-4s}\,\frac{\sqrt{5}}{s^2+\big(\sqrt{5}\big)^2}.$$

Recall that $\mathcal{L}[\sin(at)] = \dfrac{a}{(s^2+a^2)}$, then

$$\mathcal{L}[f(t)] = \frac{1}{\sqrt{5}}\,e^{-4s}\,\mathcal{L}[\sin(\sqrt{5}t)].$$

But the translation identity $e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u(t-c)f(t-c)]$ implies

$$\mathcal{L}[f(t)] = \frac{1}{\sqrt{5}}\,\mathcal{L}\big[u(t-4)\sin\big(\sqrt{5}\,(t-4)\big)\big] \quad \Rightarrow \quad f(t) = \frac{1}{\sqrt{5}}\,u(t-4)\sin\big(\sqrt{5}\,(t-4)\big).$$

$\triangleleft$

Example 6: Find the function $f(t)$ such that $\mathcal{L}[f(t)] = \dfrac{(s-1)}{(s-2)^2+3}$.

Solution: We first rewrite the right-hand side above as follows,

$$\begin{aligned}
\mathcal{L}[f(t)] &= \frac{(s-1-1+1)}{(s-2)^2+3}\\
&= \frac{(s-2)}{(s-2)^2+3} + \frac{1}{(s-2)^2+3}\\
&= \frac{(s-2)}{(s-2)^2+\big(\sqrt{3}\big)^2} + \frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{(s-2)^2+\big(\sqrt{3}\big)^2},\\
&= \mathcal{L}[\cos(\sqrt{3}\,t)](s-2) + \frac{1}{\sqrt{3}}\,\mathcal{L}[\sin(\sqrt{3}\,t)](s-2).
\end{aligned}$$

But the translation identity $\mathcal{L}[f(t)](s-c) = \mathcal{L}[e^{ct}f(t)]$ implies

$$\mathcal{L}[f(t)] = \mathcal{L}\big[e^{2t}\cos\big(\sqrt{3}\,t\big)\big] + \frac{1}{\sqrt{3}},\mathcal{L}\big[e^{2t}\sin\big(\sqrt{3}\,t\big)\big].$$

So, we conclude that

$$f(t) = \frac{e^{2t}}{\sqrt{3}}\Big[\sqrt{3}\,\cos\big(\sqrt{3}\,t\big) + \sin\big(\sqrt{3}\,t\big)\Big].$$

$\triangleleft$

3.3.3. Solving Differential Equations.

Example 7: Use the LT to find the solution to the initial IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad y(0) = 0, \qquad y'(0) = 0, \qquad b(t) = \begin{cases} 1 & 0 \leqslant t < \pi \\ 0 & t \geqslant \pi. \end{cases} \tag{3}$$

Solution: The source function b can be written as $b(t) = u(t) - u(t-\pi)$. The last expression for b is particularly useful to find its Laplace Transform,

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t-\pi)] = \frac{1}{s} + e^{-\pi s}\,\frac{1}{s} \quad \Rightarrow \quad \mathcal{L}[b(t)] = (1 - e^{-\pi s})\,\frac{1}{s}.$$

Now Laplace Transform the whole equation,

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[b].$$

Since the initial condition are $y(0) = 0$ and $y'(0) = 0$, we obtain

$$\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s} \quad \Rightarrow \quad \mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s\left(s^2 + s + \frac{5}{4}\right)}.$$

Introduce the function

$$H(s) = \frac{1}{s\left(s^2 + s + \frac{5}{4}\right)} \quad \Rightarrow \quad y(t) = \mathcal{L}^{-1}[H(s)] - \mathcal{L}^{-1}[e^{-\pi s}\,H(s)].$$

We use partial fractions to simplify H. We first find the roots of the denominator,

$$s^2 + s + \frac{5}{4} = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big],$$

so the roots are complex valued. An appropriate partial fraction decomposition is

$$H(s) = \frac{1}{s\left(s^2 + s + \frac{5}{4}\right)} = \frac{a}{s} + \frac{(bs + c)}{\left(s^2 + s + \frac{5}{4}\right)}$$

Therefore, we get

$$1 = a\left(s^2 + s + \frac{5}{4}\right) + s\,(bs + c) = (a + b)\,s^2 + (a + c)\,s + \frac{5}{4}a.$$

This equation implies that a, b, and c, satisfy the equations

$$a + b = 0, \quad a + c = 0, \quad \frac{5}{4}\,a = 1.$$

The solution is, $a = \frac{4}{5}$, $b = -\frac{4}{5}$, $c = -\frac{4}{5}$. Hence, we have found that,

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right) s} = \frac{4}{5} \Big[\frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)} \Big]$$

Complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big) s + \frac{1}{4} \Big] - \frac{1}{4} + \frac{5}{4} = \Big(s + \frac{1}{2} \Big)^2 + 1.$$

Replace this expression in the definition of H, that is,

$$H(s) = \frac{4}{5} \Big[\frac{1}{s} - \frac{(s+1)}{\left[\left(s + \frac{1}{2}\right)^2 + 1\right]} \Big]$$

Rewrite the polynomial in the numerator,

$$(s+1) = \Big(s + \frac{1}{2} + \frac{1}{2} \Big) = \Big(s + \frac{1}{2} \Big) + \frac{1}{2},$$

hence we get

$$H(s) = \frac{4}{5} \Big[\frac{1}{s} - \frac{\left(s + \frac{1}{2}\right)}{\left[\left(s + \frac{1}{2}\right)^2 + 1\right]} - \frac{1}{2} \frac{1}{\left[\left(s + \frac{1}{2}\right)^2 + 1\right]} \Big].$$

Use the Laplace Transform table to get $H(s)$ equal to

$$H(s) = \frac{4}{5} \Big[\mathcal{L}[1] - \mathcal{L}\big[e^{-t/2} \cos(t)\big] - \frac{1}{2} \mathcal{L}[e^{-t/2} \sin(t)] \Big],$$

equivalently

$$H(s) = \mathcal{L}\Big[\frac{4}{5} \Big(1 - e^{-t/2} \cos(t) - \frac{1}{2} e^{-t/2} \sin(t) \Big) \Big].$$

Denote

$$h(t) = \frac{4}{5} \Big[1 - e^{-t/2} \cos(t) - \frac{1}{2} e^{-t/2} \sin(t) \Big]. \quad \Rightarrow \quad H(s) = \mathcal{L}[h(t)].$$

Recalling $\mathcal{L}[y(t)] = H(s) + e^{-\pi s} H(s)$, we obtain $\mathcal{L}[y(t)] = \mathcal{L}[h(t)] + e^{-\pi s} \mathcal{L}[h(t)]$, that is,

$$y(t) = h(t) + u(t - \pi) h(t - \pi).$$

$\lhd$

Example 8 (Extra Example): Use the LT to find the solution to the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad y(0) = 0, \qquad y'(0) = 0, \qquad g(t) = \begin{cases} \sin(t) & 0 \leqslant t < \pi \\ 0 & t \geqslant \pi. \end{cases} \tag{4}$$

Solution: Rewrite the source function g using step functions, as follows,

$$g(t) = \big[u(t) - u(t-\pi)\big]\,\sin(t),$$

since $u(t) - u(t-\pi)$ is a box function, taking value one in the interval $[0, \pi]$ and zero on the complement. Finally, notice that the equation $\sin(t) = -\sin(t-\pi)$ implies that the function g can be expressed as follows,

$$g(t) = u(t)\,\sin(t) - u(t-\pi)\,\sin(t) \quad\Rightarrow\quad g(t) = u(t)\,\sin(t) + u(t-\pi)\,\sin(t-\pi).$$

The last expression for g is particularly useful to find its Laplace Transform,

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

With this last transform is not difficult to solve the differential equation. As usual, Laplace Transform the whole equation,

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

Since the initial condition are $y(0) = 0$ and $y'(0) = 0$, we obtain

$$\Big(s^2 + s + \frac{5}{4}\Big)\,\mathcal{L}[y] = \big(1 + e^{-\pi s}\big)\,\frac{1}{(s^2+1)} \quad\Rightarrow\quad \mathcal{L}[y] = \big(1 + e^{-\pi s}\big)\,\frac{1}{\Big(s^2 + s + \frac{5}{4}\Big)\,(s^2+1)}.$$

Introduce the function

$$H(s) = \frac{1}{\Big(s^2 + s + \frac{5}{4}\Big)\,(s^2+1)} \quad\Rightarrow\quad y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)].$$

That is, we only need to find the Inverse Laplace Transform of H. We use partial fractions to simplify the expression of H. We first find out whether the denominator has real or complex roots:

$$s^2 + s + \frac{5}{4} = 0 \quad\Rightarrow\quad s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big],$$

so the roots are complex valued. An appropriate partial fraction decomposition is

$$H(s) = \frac{1}{\left(s^2+s+\frac{5}{4}\right)\left(s^2+1\right)} = \frac{(as+b)}{\left(s^2+s+\frac{5}{4}\right)} + \frac{(cs+d)}{(s^2+1)}.$$

Therefore, we get

$$1 = (as+b)(s^2+1) + (cs+d)\Big(s^2+s+\frac{5}{4}\Big),$$

equivalently,

$$1 = (a+c)\,s^3 + (b+c+d)\,s^2 + \Big(a+\frac{5}{4}\,c+d\Big)\,s + \Big(b+\frac{5}{4}\,d\Big).$$

This equation implies that a, b, c, and d, are solutions of

$$a+c=0, \quad b+c+d=0, \quad a+\frac{5}{4}\,c+d=0, \quad b+\frac{5}{4}\,d=1.$$

Here is the solution to this system:

$$a=\frac{16}{17}, \qquad b=\frac{12}{17}, \qquad c=-\frac{16}{17}, \qquad d=\frac{4}{17}.$$

We have found that,

$$H(s) = \frac{4}{17}\Big[\frac{(4s+3)}{\left(s^2+s+\frac{5}{4}\right)} + \frac{(-4s+1)}{(s^2+1)}\Big].$$

Complete the square in the denominator,

$$s^2+s+\frac{5}{4} = \Big[s^2+2\Big(\frac{1}{2}\Big)\,s+\frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4} = \Big(s+\frac{1}{2}\Big)^2 + 1.$$

$$H(s) = \frac{4}{17}\Big[\frac{(4s+3)}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} + \frac{(-4s+1)}{(s^2+1)}\Big].$$

Rewrite the polynomial in the numerator,

$$(4s+3) = 4\Big(s+\frac{1}{2}-\frac{1}{2}\Big)+3 = 4\Big(s+\frac{1}{2}\Big)+1,$$

hence we get

$$H(s) = \frac{4}{17}\Big[4\,\frac{\left(s+\frac{1}{2}\right)}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} + \frac{1}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} - 4\,\frac{s}{(s^2+1)} + \frac{1}{(s^2+1)}\Big].$$

Use the Laplace Transform Table to get $H(s)$ equal to

$$H(s) = \frac{4}{17}\Big[4\,\mathcal{L}\big[e^{-t/2}\cos(t)\big] + \mathcal{L}\big[e^{-t/2}\sin(t)\big] - 4\,\mathcal{L}[\cos(t)] + \mathcal{L}[\sin(t)]\Big],$$

equivalently

$$H(s) = \mathcal{L}\Big[\frac{4}{17}\Big(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big)\Big].$$

Denote

$$h(t) = \frac{4}{17}\Big[4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big] \quad \Rightarrow \quad H(s) = \mathcal{L}[h(t)].$$

Recalling $\mathcal{L}[y(t)] = H(s) + e^{-\pi s}\,H(s)$, we obtain $\mathcal{L}[y(t)] = \mathcal{L}[h(t)] + e^{-\pi s}\,\mathcal{L}[h(t)]$, that is,

$$y(t) = h(t) + u(t-\pi)h(t-\pi).$$

$\triangleleft$

3.4. Generalized Sources

Section Objective(s):
- The Dirac's Delta.
- Applications and Properties.
- The Impulse Response Function.

Remarks:

- The Dirac's delta is the main example of what it is called a generalized function.
- Introduced by Paul Dirac while studying quantum mechanics.
- We want that Dirac's Delta, $\delta(t)$, to satisfy:
  - $\delta(t) = 0$ for all $t \neq 0$.
  - $\delta(0) = \infty$.
  - $\displaystyle\int_{-1}^{1} \delta(t)\, dt = 1.$

  No function has these properties.
- Dirac's delta is the limit of a sequence of functions.
- The integral of a Dirac's delta is the limit of a sequence of integrals.

3.4.1. The Dirac Delta.

Definition 1. The ***Dirac delta*** generalized function is the limit

$$\delta(t) = \lim_{n\to\infty} \delta_n(t),$$

for every fixed $t \in \mathbb{R}$ of the sequence functions $\{\delta_n\}_{n=1}^{\infty}$,

$$\delta_n(t) = n\Big[u(t) - u\big(t - \frac{1}{n}\big)\Big].$$

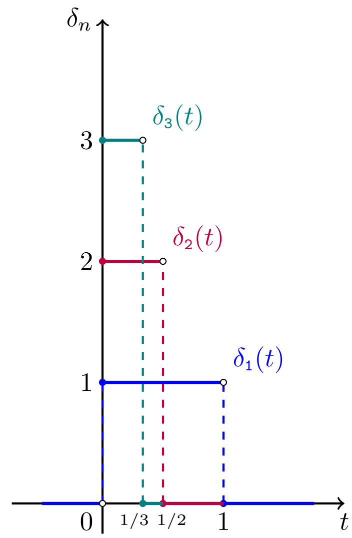
Remark: The sequence of bump functions introduced above can be rewritten as follows,

$$\delta_n(t) = \begin{cases} 0, & t < 0 \\ n, & 0 \leqslant t < \dfrac{1}{n} \\ 0, & t \geqslant \dfrac{1}{n}. \end{cases}$$

We then obtain the equivalent expression,

$$\delta(t) = \begin{cases} 0 & \text{for} \quad t \neq 0, \\ \infty & \text{for} \quad t = 0. \end{cases}$$

Remark: There are infinitely many sequences $\{\delta_n\}$ of functions with the Dirac delta as their limit as $n \to \infty$.



Interactive Graph: Dirac's Delta.

Remarks:

(a) The Dirac delta is the function zero on the domain $\mathbb{R} - \{0\}$.

(b) The Dirac delta is not a function on $\mathbb{R}$.

(c) We define: $\displaystyle\int_{-1}^{1} \delta(t)\,dt = \lim_{n\to\infty} \int_{-1}^{1} \delta_n(t)\,dt.$

Theorem 1.

$$\int_{-1}^{1} \delta(t)\,dt = 1.$$

3.4.2. Applications and Properties.

Applications:

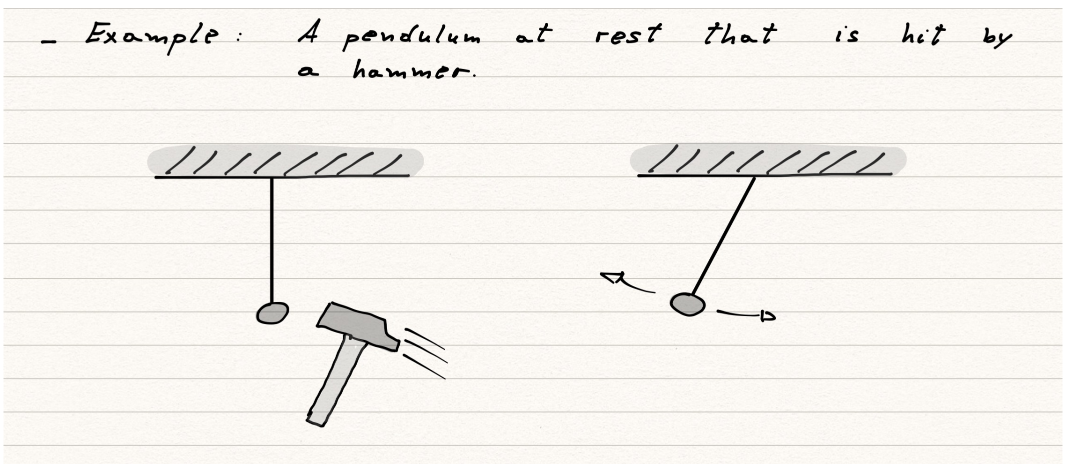
(a) Dirac's delta generalized function is useful to describe

impulsive forces.

(b) An impulsive force transfers a ***finite momentum***

in an ***infinitely short time***.

(c) For example, a pendulum at rest that is hit by a hammer.



Main Properties:

Theorem 2. If f is continuous on (a, b) and $c \in (a, b)$, then

$$\int_a^b f(t)\,\delta(t-c)\,dt = f(c).$$

Proof of Theorem 2: We again compute the integral of a Dirac's delta as a limit of a sequence of integrals,

$$\begin{aligned}
\int_a^b \delta(t-c)\,f(t)\,dt &= \lim_{n\to\infty} \int_a^b \delta_n(t-c)\,f(t)\,dt \\
&= \lim_{n\to\infty} \int_a^b n\Big[u(t-c) - u\big(t-c-\frac{1}{n}\big)\Big]\,f(t)\,dt \\
&= \lim_{n\to\infty} \int_c^{c+\frac{1}{n}} n\,f(t)\,dt, \qquad \frac{1}{n} < (b-c).
\end{aligned}$$

To get the last line we used that $c \in [a,b]$. Let F be any primitive of f, so $F(t) = \int f(t)\,dt$. Then we can write,

$$\begin{aligned}
\int_a^b \delta(t-c)\,f(t)\,dt &= \lim_{n\to\infty} n\big[F\big(c+\frac{1}{n}\big) - F(c)\big] \\
&= \lim_{n\to\infty} \frac{1}{\big(\frac{1}{n}\big)}\big[F\big(c+\frac{1}{n}\big) - F(c)\big], \qquad h = \frac{1}{n}, \\
&= \lim_{h\to 0} \frac{F(c+h) - F(c)}{h} \\
&= F'(c) \\
&= f(c).
\end{aligned}$$

This establishes the Theorem.

$\square$

Theorem 3. For all $s \in \mathbb{R}$ holds

$$\mathcal{L}[\delta(t-c)] = \begin{cases} \underline{e^{-cs}\qquad\qquad} & \text{for} \quad c \geqslant 0, \\ \underline{0\qquad\qquad\quad} & \text{for} \quad c < 0. \end{cases}$$

Proof of Theorem 4.4.5: We use the previous Theorem on the integral that defines a Laplace transform,

$$\mathcal{L}[\delta(t-c)] = \int_0^\infty e^{-st}\,\delta(t-c)\,dt = \begin{cases} e^{-cs} & \text{for} \quad c \geqslant 0, \\ 0 & \text{for} \quad c < 0, \end{cases}$$

This establishes the Theorem.

$\square$

Example 1: Find the solution y to the initial value problem

$$y'' + y = \delta(t-3), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: The source is a generalized function, so we need to solve this problem using the Laplace Transform. So we compute the Laplace Transform of the differential equation,

$$\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[\delta(t-3)] \quad \Rightarrow \quad (s^2+1)\,\mathcal{L}[y] = e^{-3s},$$

where in the second equation we have already introduced the initial conditions $y(0) = 0$, $y'(0) = 0$. We arrive to the equation

$$\mathcal{L}[y] = e^{-3s}\,\frac{1}{(s^2+1)}$$

$$\mathcal{L}[y] = e^{-3s}\,\mathcal{L}[\sin(t)]$$

Recalling the translation identity

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u(t-c)\,f(t-c)],$$

we get that

$$\mathcal{L}[y] = \mathcal{L}[u(t-3)\,\sin(t-3)],$$

which leads to the solution

$$y(t) = u(t-3)\,\sin(t-3).$$

$\lhd$

3.4.3. The Impulse Response Function.

Definition 2. The ***impulse response function*** at the point $c \geqslant 0$ of the linear operator

$$L(y) = y'' + a_1\, y' + a_0\, y,$$

with a_1, a_0 constants, is the solution y_δ of

$$L(y_\delta) = \delta(t-c), \qquad y_\delta(0) = 0, \qquad y'_\delta(0) = 0.$$

Theorem 4. The function y_δ is the impulse response function at $c \geqslant 0$ of the constant coefficients operator $L(y) = y'' + a_1\, y' + a_0\, y$ iff holds

$$y_\delta = \mathcal{L}^{-1}\Big[\frac{e^{-cs}}{p(s)}\Big].$$

where p is the characteristic polynomial of L.

Proof of Theorem 4: Compute the Laplace transform of the differential equation for for the impulse response function y_δ,

$$\mathcal{L}[y''] + a_1\, \mathcal{L}[y'] + a_0\, \mathcal{L}[y] = \mathcal{L}[\delta(t-c)] = e^{-cs}.$$

Since the initial data for y_δ is trivial, we get

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y] = e^{-cs}.$$

Since $p(s) = s^2 + a_1 s + a_0$ is the characteristic polynomial of L, we get

$$\mathcal{L}[y] = \frac{e^{-cs}}{p(s)} \quad \Leftrightarrow \quad y(t) = \mathcal{L}^{-1}\Big[\frac{e^{-cs}}{p(s)}\Big].$$

We notice that all the steps in this calculation are if and only ifs. This establishes the Theorem.

$\square$

Example 2 (Extra Example): Use the Laplace Transform to show that the solutions to the IVP below are the same, where

$$y'' + a_1 y' + a_0 y = \delta(t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

and

$$y'' + a_1 y' + a_0 y = 0, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Provide a physics-based explanation of why these solutions coincide.

Solution:

For the first IVP we have

$$(s^2 + a_1\, s + a_0)\, \mathcal{L}[y] = \mathcal{L}[\delta(t)] = 1 \quad \Rightarrow \quad \mathcal{L}[y] = \frac{1}{(s^2 + a_1\, s + a_0)}.$$

For the second IVP we have

$$(s^2 \mathcal{L}[y] - s\, y(0) - y'(0)) + a_1 (s\mathcal{L}[y] - y(0)) + a_0\, \mathcal{L}[y] = 0,$$

but the initial conditions imply

$$s^2 \mathcal{L}[y] - 1 + a_1\, s\mathcal{L}[y] + a_0\, \mathcal{L}[y] = 0,$$

so we get

$$(s^2 + a_1\, s + a_0)\, \mathcal{L}[y] = 1 \quad \Rightarrow \quad \mathcal{L}[y] = \frac{1}{(s^2 + a_1\, s + a_0)}.$$

The action of the impulsive force is to produce a nonzero velocity at the initial time, because it transfers a nontrivial momentum at a single point in time.

$\triangleleft$

3.5. Convolutions and Solutions

Section Objective(s):
- The Convolution of Two Functions.
- Main Properties of the Convolution.
- The Solution Decomposition Theorem.

Remarks:

- We introduce a new operation between two function, the convolution.
- The convolution is a nonlocal product of two functions.
- We know that $\mathcal{L}[fg] \neq \mathcal{L}[f]\,\mathcal{L}[g]$.
- The convolution of f, g is the function such that

$$\mathcal{L}\big[\text{Convolution}(fg)\big] = \mathcal{L}[f]\,\mathcal{L}[g].$$

- The convolution is defined for the Dirac's delta.
- The Dirac's delta is the identity element for the convolution.

3.5.1. The Convolution of Two Functions.

Definition 1. The ***convolution*** of functions f and g is a function $f * g$ given by

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

Remark: The convolution is defined even when either f or g is a Dirac's delta.

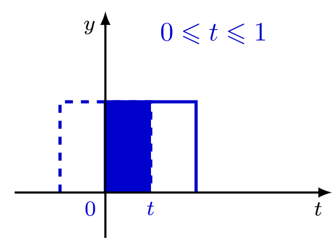
Example 1: Find $f * g$ the convolution of the functions $f(t) = b(t)$ and $g(t) = b(t)$, where we denoted $b(t) = u(t) - u(t - 1)$, the bump function on $[0, 1]$.

Interactive Graph: Convolution of Bumps

Solution: The definition of convolution is,

$$(b * b)(t) = \int_0^t b(\tau) b(t - \tau)\, d\tau.$$

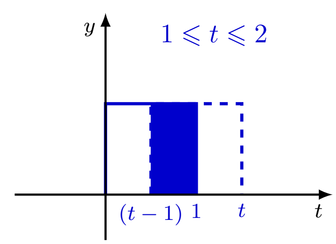
- $b(\tau) = 1$ for $\tau \in [0, 1]$ and is zero otherwise.
- $b(t - \tau) = 1$ for



$$0 \leqslant t - \tau \leqslant 1 \quad \Rightarrow \quad -t \leqslant -\tau \leqslant 1 - t \quad \Rightarrow \quad t \geqslant \tau \geqslant -1 + t,$$

so $\tau \in [t - 1, t]$ and is zero otherwise.

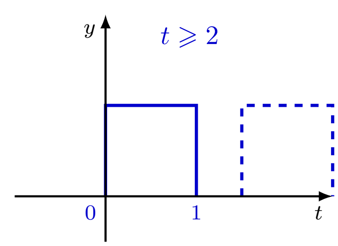
- For $t \in [0, 1]$, we have



$$(b * b)(t) = \int_0^t 1\, dt = t.$$

For $t \in [1, 2]$ we have

$$(b * b)(t) = \int_{t-1}^1 1\, d\tau = 1 - (t - 1) = 2 - t.$$



For $t \in [2, \infty)$, $(b * b)(t) = 0$. We then conclude that

$$(f * g)(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ 2 - t, & 1 < t \leq 2, \\ 0, & t > 2. \end{cases}$$

◁

Example 2: Graph the convolution of

$$f(\tau) = u(\tau) - u(\tau - 1),$$
$$g(\tau) = \begin{cases} 2\,e^{-2\tau} & \text{for} \quad \tau \geqslant 0 \\ 0 & \text{for} \quad \tau < 0. \end{cases}$$

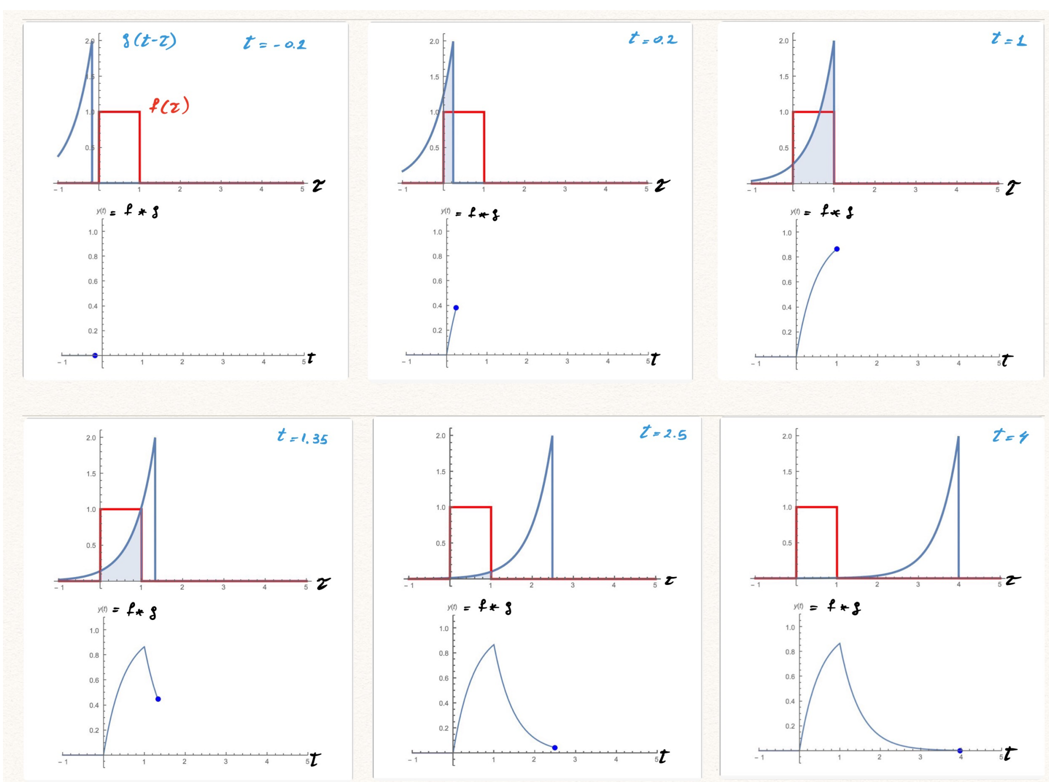
Interactive Graph: Convolution of Bump and Exponential (Slow)

Solution: Notice that

$$g(-\tau) = \begin{cases} 2\,e^{2\tau} & \text{for} \quad \tau \leqslant 0 \\ \\ 0 & \text{for} \quad \tau > 0. \end{cases}$$

Then we have that

$$g(t-\tau) = g(-(\tau - t)) \begin{cases} 2\,e^{2(\tau - t)} & \text{for} \quad \tau \leqslant t \\ \\ 0 & \text{for} \quad \tau > t. \end{cases}$$



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3.5.2. Main Properties of the Convolution.

Theorem 1. (Laplace Transform) If $\mathcal{L}[f]$ and $\mathcal{L}[g]$, exist, then

$$\mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g].$$

Remark: This is the origin of the convolution operation. Since

$$\mathcal{L}[f]\,\mathcal{L}[g] \neq \mathcal{L}[f g],$$

people were interested in finding a function h such that

$$\mathcal{L}[f]\,\mathcal{L}[g] = \mathcal{L}[h].$$

The answer is, $h = f * g$

Idea of the Proof: Switch the order of the integrals.

Other Properties of Convolutions:

Theorem 2. For every piecewise continuous functions f, g, and h, hold:

(i) Commutativity: $f * g = g * f$;

(ii) Associativity: $f * (g * h) = (f * g) * h$;

(iii) Distributivity: $f * (g + h) = f * g + f * h$;

(iv) Neutral element: $f * 0 = 0$;

(v) Identity element: $f * \delta = f$.

Example 3: Find the function g such that $f(t) = \int_0^t \sin(4\tau)\, g(t-\tau)\, d\tau$ has the Laplace transform $\mathcal{L}[f] = \dfrac{s}{(s^2+16)((s-1)^2+9)}$.

Solution: Since $f(t) = \sin(4t) * g(t)$, we can write

$$\frac{s}{(s^2+16)((s-1)^2+9)} = \mathcal{L}[f] = \mathcal{L}[\sin(4t) * g(t)] = \mathcal{L}[\sin(4t)]\, \mathcal{L}[g] = \frac{4}{(s^2+4^2)}\, \mathcal{L}[g],$$

so we get that

$$\frac{4}{(s^2+4^2)}\, \mathcal{L}[g] = \frac{s}{(s^2+16)((s-1)^2+9)} \quad \Rightarrow \quad \mathcal{L}[g] = \frac{1}{4}\, \frac{s}{(s-1)^2+3^2}.$$

We now rewrite the right-hand side of the last equation,

$$\mathcal{L}[g] = \frac{1}{4}\, \frac{(s-1+1)}{(s-1)^2+3^2} \quad \Rightarrow \quad \mathcal{L}[g] = \frac{1}{4}\Big(\frac{(s-1)}{(s-1)^2+3^2} + \frac{1}{3}\, \frac{3}{(s-1)^2+3^2}\Big),$$

that is,

$$\mathcal{L}[g] = \frac{1}{4}\Big(\mathcal{L}[\cos(3t)](s-1) + \frac{1}{3}\mathcal{L}[\sin(3t)](s-1)\Big) = \frac{1}{4}\Big(\mathcal{L}[e^t\, \cos(3t)] + \frac{1}{3}\, \mathcal{L}[e^t\, \sin(3t)]\Big),$$

which leads us to

$$g(t) = \frac{1}{4}\, e^t\, \Big(\cos(3t) + \frac{1}{3}\, \sin(3t)\Big)$$

$\lhd$

3.5.3. The Solution Decomposition Theorem.

Theorem 3. (Solution Decomposition) The solution of

$$L(y) = g(t), \quad y(0) = y_0, \quad y'(0) = y_1,$$

where $L(y) = y'' + a_1\, y' + a_0\, y$ has constant coefficients, can be decomposed as

$$y(t) = y_h(t) + (y_\delta * g)(t),$$

where y_h is the solution of the homogeneous initial value problem

$$L(y_h) = 0, \quad y_h(0) = y_0, \quad y_h'(0) = y_1,$$

and y_δ is the impulse response function ____________________ of L.

Remarks:

(1) The solution decomposition above can be written in the equivalent way

$$y(t) = y_h(t) + \int_0^t y_\delta(\tau)\, g(t-\tau)\, d\tau.$$

(2) Recall that the impulse response function is the solution of

$$L(y_\delta) = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

(3) Recall that the impulse response function can be written as

$$y_\delta = \mathcal{L}^{-1}\Big[\frac{1}{p(s)}\Big].$$

Example 4: Use the Solution Decomposition Theorem to express the solution of
$$y'' + 2\,y' + 2\,y = g(t), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: We first find the impuse response function
$$y_\delta(t) = \mathcal{L}^{-1}\Big[\frac{1}{p(s)}\Big], \qquad p(s) = s^2 + 2s + 2.$$
since p has complex roots, we complete the square,
$$s^2 + 2s + 2 = s^2 + 2s + 1 - 1 + 2 = (s+1)^2 + 1,$$
so we get
$$y_\delta(t) = \mathcal{L}^{-1}\Big[\frac{1}{(s+1)^2 + 1}\Big] \quad \Rightarrow \quad y_\delta(t) = e^{-t}\,\sin(t).$$
We now compute the solution to the homogeneous problem
$$y_h'' + 2\,y_h' + 2\,y_h = 0, \qquad y_h(0) = 1, \quad y_h'(0) = -1.$$
Using Laplace transforms we get
$$\mathcal{L}[y_h''] + 2\,\mathcal{L}[y_h'] + 2\,\mathcal{L}[y_h] = 0,$$
and recalling the relations between the Laplace transform and derivatives,
$$\big(s^2\,\mathcal{L}[y_h] - s\,y_h(0) - y_h'(0)\big) + 2\big(\mathcal{L}[y_h'] = s\,\mathcal{L}[y_h] - y_h(0)\big) + 2\mathcal{L}[y_h] = 0,$$
using our initial conditions we get $(s^2 + 2s + 2)\,\mathcal{L}[y_h] - s + 1 - 2 = 0$, so
$$\mathcal{L}[y_h] = \frac{(s+1)}{(s^2 + 2s + 2)} = \frac{(s+1)}{(s+1)^2 + 1} \quad \Rightarrow \quad y_h(t) = \mathcal{L}\Big[e^{-t}\,\cos(t)\Big].$$
Therefore, the solution to the original initial value problem is
$$y(t) = y_h(t) + (y_\delta * g)(t) \quad \Rightarrow \quad y(t) = e^{-t}\,\cos(t) + \int_0^t e^{-\tau}\,\sin(\tau)\,g(t-\tau)\,d\tau.$$

$\lhd$

Proof of Theorem 1:

$$\begin{aligned}
\mathcal{L}[f]\,\mathcal{L}[g] &= \Big[\int_0^\infty e^{-st} f(t)\,dt\Big]\,\Big[\int_0^\infty e^{-st_1} g(t_1)\,dt_1\Big] \\
&= \int_0^\infty e^{-st_1} g(t_1)\Big(\int_0^\infty e^{-st} f(t)\,dt\Big)\,dt_1 \\
&= \int_0^\infty g(t_1)\Big(\int_0^\infty e^{-s(t+t_1)} f(t)\,dt\Big)\,dt_1,
\end{aligned}$$

Change of variables in the inside integral $\tau = t + t_1$, hence $d\tau = dt$. Then, we get

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(t_1)\Big(\int_{t_1}^\infty e^{-s\tau} f(\tau - t_1)\,d\tau\Big)\,dt_1 = \int_0^\infty \int_{t_1}^\infty e^{-s\tau}\, g(t_1)\, f(\tau - t_1)\,d\tau\,dt_1. \quad (1)$$

Here is the key step. We must switch the order of integration.

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau}\, g(t_1)\, f(\tau - t_1)\,dt_1\,d\tau.$$

Then, is straightforward to check that

$$\begin{aligned}
\mathcal{L}[f]\,\mathcal{L}[g] &= \int_0^\infty e^{-s\tau}\,\Big(\int_0^\tau g(t_1)\, f(\tau - t_1)\,dt_1\Big)\,d\tau \\
&= \int_0^\infty e^{-s\tau}(g * f)(\tau)\,dt \\
&= \mathcal{L}[g * f]
\end{aligned}$$

So we conclude that

$$\mathcal{L}[f]\,\mathcal{L}[g] = \mathcal{L}[f * g].$$

$\square$

4.1. Modeling with First Order Systems

Section Objective(s):
- Interacting Species.
- Predator-Prey.
- Spring-Mass as a First Order System.
- Equilibrium Solutions.

Remarks:

- We have studied how to solve several first order ________ equations

$$y' = f(t, y).$$

- We have also student how to solve second order linear ________ equations

$$y'' + a_1\, y' + a_0\, y = f(t).$$

- There are more complex physical system that cannot ________ be decribed with the equations above.
- Today we see two of such systems: interacting species ________ and predator- prey ________ systems.
- Then we see that second order ________ equations can be written as first order ________ systems.

4.1.1. Review: Interacting Species.

Example 1: Construct a differential equation that describes the population of rabbits and sheep coexisting in an environment with finite resources.

Solution:

- First assume that there are only rabbits, **unlimited** resources.

$$R'(t) = r_R\, R(t), \qquad r_R \quad \text{growth rate coefficient, rabbits.}$$

- Now, assume only rabbits, **limited** resources.

$$R'(t) = r_R\, R(t)\Big(1 - \frac{R(t)}{K_R}\Big), \qquad K_R \quad \text{carrying capacity, rabbits.}$$

- Assume we have rabbits and sheeps, each with limited resources, **not interacting**; for example they eat different foods.

$$R'(t) = r_R\, R(t)\Big(1 - \frac{R(t)}{K_R}\Big),$$
$$S'(t) = r_S\, S(t)\Big(1 - \frac{S(t)}{K_S}\Big).$$

- Finally, we add the interaction. These species **compete**.

$$R'(t) = r_R\, R(t)\Big(1 - \frac{R(t)}{K_R}\Big) - c_1\, R(t)\, S(t),$$
$$S'(t) = r_S\, S(t)\Big(1 - \frac{S(t)}{K_S}\Big) - c_2\, R(t)\, S(t),$$

  where $c_1 > 0$, $c_2 > 0$ are the competing coefficients. The negative sign means both R' and S' **decrease** because of the interaction.

- The product $R(t)\, S(t)$ is a simple measure of how often the two population meet.

$\triangleleft$

Definition 1. The ***interacting species system*** for the functions x and y, which depend on the independent variable t, is

$$x' = r_x\,x\Big(1 - \frac{x}{x_c}\Big) + \alpha\,x\,y$$
$$y' = r_y\,y\Big(1 - \frac{y}{y_c}\Big) + \beta\,x\,y,$$

where the constants r_x, r_y and x_c, x_c are positive and α, β are real numbers. Furthermore, we have the following particular cases:

- The species compete iff $\alpha < 0$ and $\beta < 0$.
- The species cooperate iff $\alpha > 0$ and $\beta > 0$.
- The species y cooperates with x when $\alpha > 0$,

  and x competes with y when $\beta < 0$.

Example 2: The interaction of of rabbits and elephants is given by

$$x'(t) = \frac{1}{2}\,x(t) - \frac{1}{20}\,x^2(t) - x(t)\,y(t),$$
$$y'(t) = 3\,y(t) - \frac{1}{300}\,y^2(t) - 200\,x(t)\,y(t),$$

which variable represents the elephants? What is the growth rate and carrying capacity of the elephants and of the rabbits?

Solution:

- The x grow slower than the y, since $r_x = 1/2$ while $r_y = 3$. So x are elephants.
- Since

  $$x'(t) = \frac{1}{2}\,x(t) - \frac{1}{20}\,x^2(t) - x(t)\,y(t) = \frac{1}{2}\,x(t)\Big(1 - \frac{1}{\frac{1}{2}20}\,x(t)\Big) - x(t)\,y(t),$$

  the carrying capacity $K_x = 10$. Analogously $K_y = 900$ since

  $$y'(t) = 3\,y(t) - \frac{1}{300}\,y^2(t) - 200\,x(t)\,y(t) = 3\,y(t)\Big(1 - \frac{1}{3(300)}\,y(t)\Big) - 200\,x(t)\,y(t),$$

  So x are elephants and y are rabbits, the same enviroment supports 10 elephants and 900 rabbits.
- The interaction affects x' by a factor -1, and y' by a factor 200. Then x are elephants, they are much less affected than rabbits by the interaction. $\triangleleft$

4.1.2. Predator-Prey System.

Example 3: Construct a differential equation that describes the population of rabbits and foxes coexisting in an environment with unlimited resources for the rabbits.

Solution:

- First assume that there are only rabbits, **unlimited** resources.

$$R'(t) = r_R\, R(t), \qquad r_R \quad \text{growth rate coefficient, rabbits.}$$

- Now, assume that there are only foxes, no rabbits. Foxes **only eat** rabbits, so **no resources**.

$$F'(t) = -r_F\, F(t), \qquad r_F \quad \text{death rate coefficient, rabbits.}$$

- Assume that the foxes meet the rabbits and eat them. This interaction has two effects:
  - The rabbit populations **decreases** its growth rate: R' is **smaller** than without the interaction.
  - The foxes populations **increases** its growth rate: F' is **larger** than without the interaction.

  Since $R(t)\, F(t)$ is a simple measure of how often the two populations meet,

$$R'(t) = r_R\, R(t) - d_1\, R(t)\, F(t),$$

$$F'(t) = -r_F\, F(t) + d_2\, R(t)\, F(t),$$

◁

Definition 2. The ***predator-prey system*** for the predator function x and the prey function y, which depend on the independent variable t, is

$$x' = -a_x\, x + b_x\, x\, y$$
$$y' = a_y\, y - b_y\, x\, y,$$

where the coefficients a_x, b_x, a_y, and b_y are nonnegative.

Remark:

- A predator is called lethargic if they seldom catch prey but can live for a long time on a single prey, for example boa constrictors.
- A predator is called active if they catch prey very often and they can live for only a short time on a single prey, for example bobcats.

Example 4: Identify which of the systems below corresponds to a lethargic predator and which one to an active predator.

$$x' = 0.3\,x - 0.1\,xy, \qquad \tilde{x}' = 0.3\,\tilde{x} - 3\,\tilde{x}\tilde{y}$$
$$y' = -0.1\,y + 2\,xy, \qquad \tilde{y}' = -2\,\tilde{y} + 0.1\,\tilde{x}\tilde{y}.$$

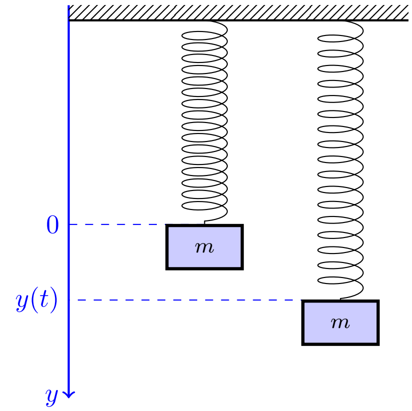
Solution:

- The variables y and $\tilde{y}$ are the predators.
- The increase in y' for eating a prey is largend than the increase in $\tilde{y}$.
- So, y benefits more froma single prey than $\tilde{y}$.
- So, y are snakes, and $\tilde{y}$ are bobcats.

◁

4.1.3. Spring-Mass as a First Order System.

Example 5. (Mass-Spring System): Consider an object of mass m mass hanging at the bottom of a spring with spring constant k, and moving in a fluid with damping constant d. Assume that there is an external force f, which depends on t, acting on the object.



If $y(t)$ is the displacement from the equilibrium position at the time t, positive downwards, the equation of motion for the variable y is

$$m\,y'' + d\,y' + k\,y = f(t).$$

Write the differential equation above as a first order system.

Solution:

- Introduce the variables $x_1 = y$ and $x_2 = y'$.
- This definitions implies that these variables are related:

$$x_1' = y' = x_2 \quad \Rightarrow \quad x_1' = x_2.$$

- The equation for x_2' is the following,

$$x_2' = y'' = -\frac{d}{m}\,y' - \frac{k}{m}\,y + f = -\frac{d}{m}\,x_2 - \frac{k}{m}\,x_1 + f,$$

  that is,

$$x_2' = -\frac{k}{m}\,x_1 - \frac{d}{m}\,x_2 + f.$$

- So we obtained the first order system

$$x_1' = x_2$$
$$x_2' = -\frac{k}{m}\,x_1 - \frac{d}{m}\,x_2 + f.$$

- This system is first order, 2×2, and linear. After our review of Linear Algebra,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

$\lhd$

4.1.4. Equilibrium Solutions.

Remark: Equilibrium solutions are defined for autonomous ssystems

Definition 3. (Equilibrium Solutions) The equilibrium solutions of the autonomous system

$$x' = f(\cancel{t}, x, y)$$
$$y' = g(\cancel{t}, x, y),$$

are solutions of the form $x(t) = x_0,\ y(t) = y_0$, for all t,

which satisfy $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$.

The point (x_0, y_0) in the xy-plane is called an equilibrium point of the system.

Example 6: Find the equilibrium solutions of the following competing species system.

$$R' = 3R\,(1 - S - R)$$
$$S' = 2S\,(2 - S - 3R)\,.$$

Solution:

- The equation for the equilibrium solutions are

$$\left.\begin{aligned} 3R\,(1 - S - R) &= 0 \\ 2S\,(2 - S - 3R) &= 0. \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} R = 0 \quad \text{and} \quad S = 0, \\ R = 0 \quad \text{and} \quad S = 2, \\ S = 0 \quad \text{and} \quad R = 1, \end{cases}$$

  so we get the points $(0, 0)$, $(0, 2)$, $(1, 0)$.

- In addition we could have

$$\left.\begin{aligned} 1 - H - R &= 0 \\ 2 - H - 3R &= 0, \end{aligned}\right\} \quad \Rightarrow \quad 1 - 2R = 0 \quad \Rightarrow \quad R = \frac{1}{2} \quad \Rightarrow \quad H = \frac{1}{2}$$

  which yields an additional equilibrium point, $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.

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Example 7. (Extra Example): Now consider a particular example of a predator-prey system.

$$\frac{dR}{dt} = 5R - 2RF$$
$$\frac{dF}{dt} = -F + 0.5RF$$

Find the equilibrium solutions of the system. Discuss what they represent in the physical situation of the model.

Solution:

- We need to solve

$$5R - 2RF = 0$$

$$-F + 0.5RF = 0.$$

Equivalently,

$$\left.\begin{aligned} R(5-2F) &= 0 \\ F(-1+0.5R) &= 0. \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} R = 0 \quad \text{and} \quad F = 0, \\ R = 2 \quad \text{and} \quad F = \dfrac{5}{2}, \end{cases}$$

- thus, the equilibrium points are $(R_0, F_0) = (0, 0)$ and $(R_1, F_1) = (2, 5/2)$.
- If $R(t) = 0$ and $F(t) = 0$, i.e. there are no rabbits and no foxes, the system is in perfect balance (their numbers will not change).
- Similarly, $R(t) = 2$ and $F(t) = 2.5$ for all $t \geq 0$ is a solution to the system of differential equations that does not change with time - the system is in balance.

$\triangleleft$

4.2. Vector Fields and Qualitative Solutions

Section Objective(s):
- Vector and Direction Fields.
- Phase Portraits and Solution Curves.

Remarks:

- There are formulas for solutions of first order systems of differential equations.
- But there are not such formulas for solutions of first order, nonlinear systems.
- It is important to find qualitative properties of solutions to nonlinear systems without solving the system.
- Qualitative graphs of solutions can be obtained from the equation, without solving the equation.
- One need to plot the vector field of the equation.

4.2.1. Vector and Direction Fields.

Definition 1. (Vector Field) The ***vector field*** of the autonomous system

$$x' = f(\cancel{t}, x, y)$$

$$y' = g(\cancel{t}, x, y),$$

is the collection of vectors $\boldsymbol{F}(x, y) = \langle f(x, y), g(x, y)\rangle$

at points (x, y) in the $xy-$plane.

Remark: The vector field

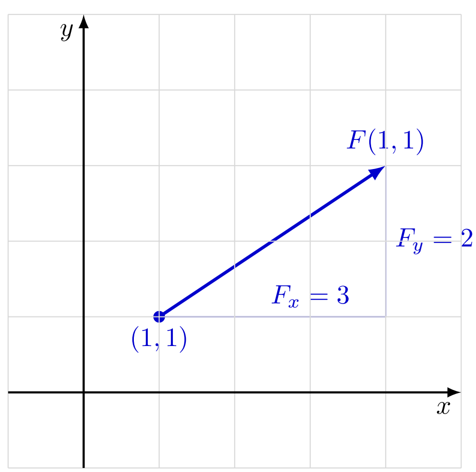
$$\boldsymbol{F}(x, y) = \langle F_x, F_y\rangle = \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

is a vector with origin at (x, y), which has horizontal component F_x and vertical component F_y.

Example 1: If the vector field is $\boldsymbol{F}(x, y) = \langle x + 2y, 4x - 2y\rangle$, draw the vector $\boldsymbol{F}(1, 1)$.

Solution:

We first compute $\boldsymbol{F}(1, 1) = \langle 1 + 2, 5 - 2\rangle = \langle 3, 2\rangle$, so $\boldsymbol{F}(1, 1) = \langle 3, 2\rangle = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.



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Example 2: Consider an object of mass $m = 1$ hanging from a pring with spring constant $k = 1$, oscillating in the air.

(1) Write the differential equation for this mass-spring system as a first order system.

(2) Draw the vector field of this system at several points on the plane.

Solution:

Part (1) The second order equation for the variable $y(t)$ is

$$y'' + y = 0.$$

We introduce $x_1 = y$ and $x_2 = y'$.

Remark: Instead of x and y we use the variables x_1 and x_2. Similarly, we denote vector fields as $\boldsymbol{F}(x_1, x_2) = \langle F_1, F_2 \rangle = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$

Then we get the system

$$x_1' = x_2$$

$$x_2' = -x_1.$$

The **vector field** is $\boldsymbol{F}(x_1, x_2) = \langle x_2, -x_1 \rangle = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$.

Part (2) Let us plot the vector field $\boldsymbol{F}(x_1, x_2)$ at various points in the x_1x_2−plane:

$\mathbf{F}(1, 0) = \langle 0, -1 \rangle$

$\mathbf{F}(0, 1) = \langle 1, 0 \rangle$

$\mathbf{F}(-1, 0) = \langle 0, 1 \rangle$

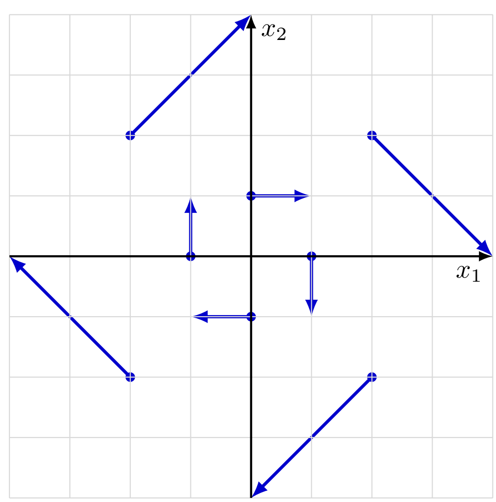
$\mathbf{F}(0, -1) = \langle -1, 0 \rangle$

$\mathbf{F}(2, 2) = \langle 2, -2 \rangle$

$\mathbf{F}(2, -2) = \langle -2, -2 \rangle$

$\mathbf{F}(-2, -2) = \langle -2, 2 \rangle$

$\mathbf{F}(-2, 2) = \langle 2, 2 \rangle$



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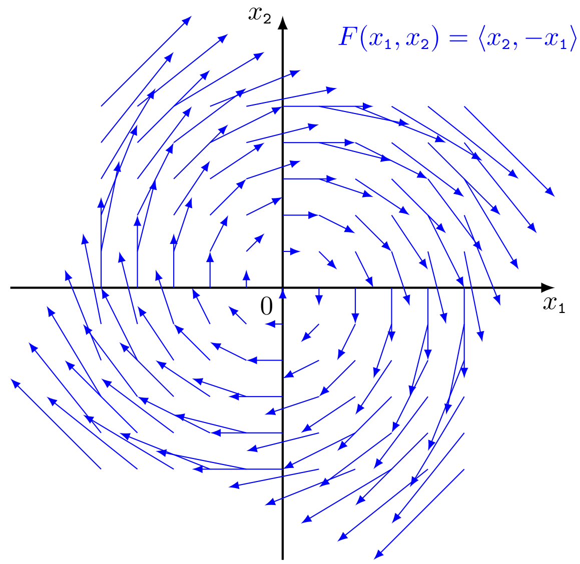


FIGURE 6. The vector field $F(x_1, x_2) = \langle x_2, -x_1 \rangle$ associated to the equation $x_1' = x_2$, $x_2' = -x_1$ which is the first order system of $y'' + y = 0$.

Remark: A direction field is a normalized version of a vector field. All the vectors have length one.

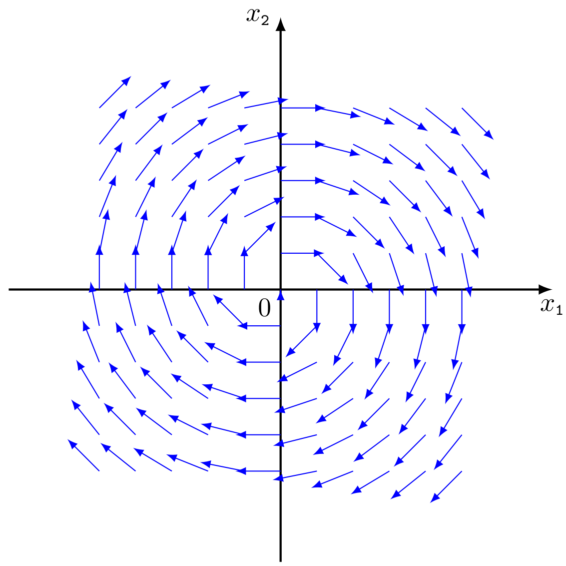


FIGURE 7. Direction field of the vector field $F(x_1, x_2) = \langle x_2, -x_1 \rangle$.

Example 3: Match each of the following direction fields to one of the systems below.

$x' = x - 1$	$x' = -y$	$x' = -1 + y$	$x' = -y$
$y' = y - 1$	$y' = x$	$y' = 1 - x$	$y' = -x$
Fig. 2	Fig. 1	Fig. 4	Fig. 3

Hint: (1) Compute the equilibrium points. (2) Evaluate the field along a subset.

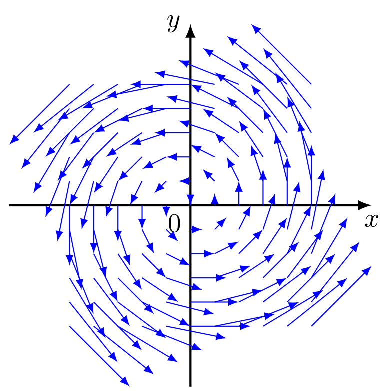


Figure 1.

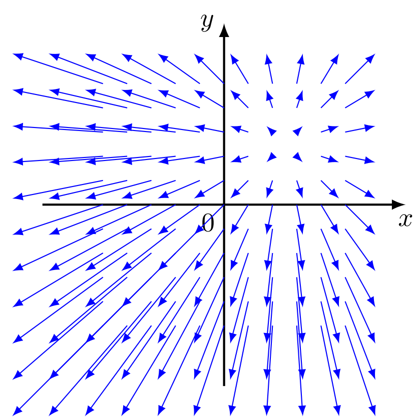


Figure 2.

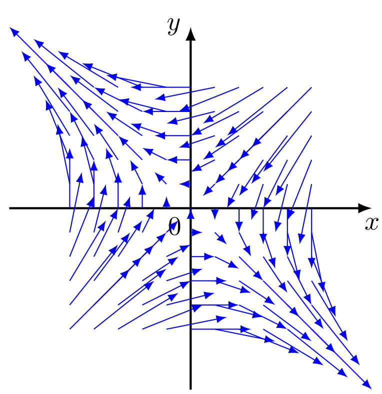


Figure 3.

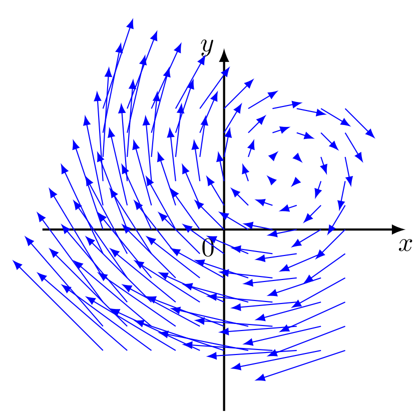


Figure 4.

4.2.2. Phase Portraits and Solution Curves.

Remark: Direction and Vector fields are useful to draw solution curves.

The resulting picture is called a phase portrait.

Theorem 1. If $x(t)$ and $y(t)$ are solutions of the autonomous differential system

$$x'(t) = f(x(t), y(t))$$
$$y'(t) = g(x(t), y(t)),$$

then the solution curve $r(t) = (x(t), y(t))$ on the xy-plane is tangent to the vector field $\boldsymbol{F}(x, y) = \langle f, g \rangle$, that is,

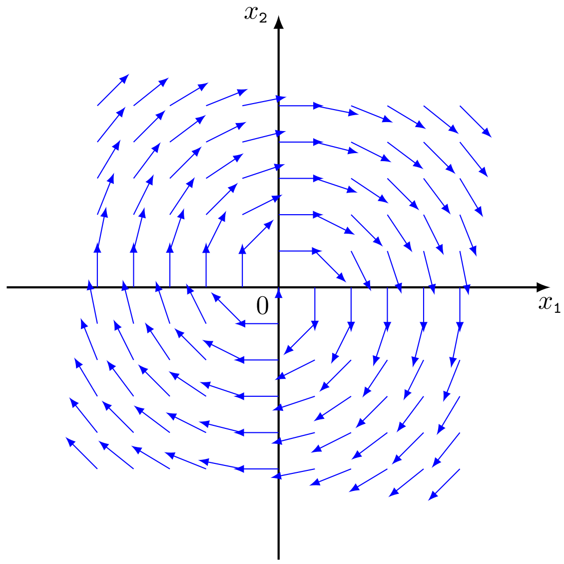
$$\boldsymbol{r}'(t) = \boldsymbol{F}(r(t)).$$

Example 4: Use the direction field of the mass-spring system to draw qualitative graphs of its solutions.

Solution:

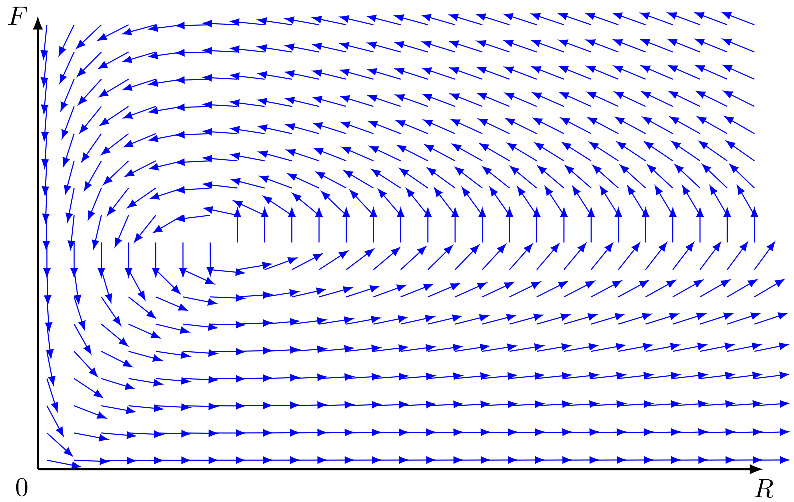
The equation is

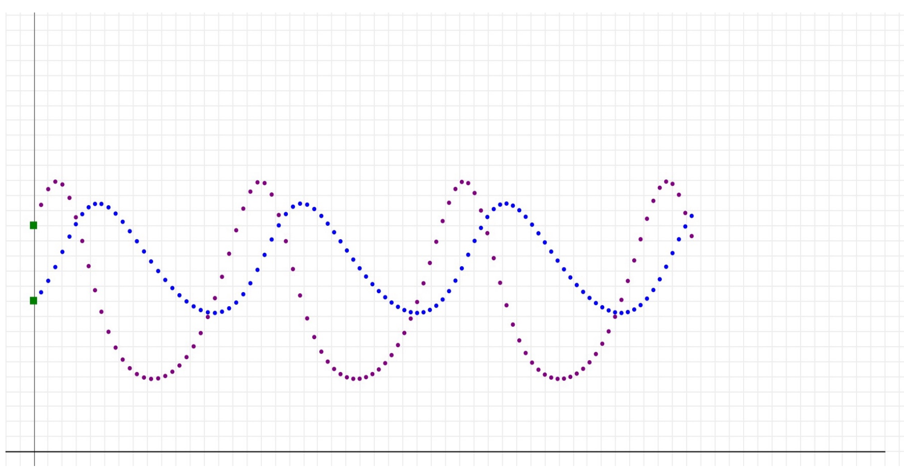
$$y'' + y = 0 \quad \Rightarrow \quad \begin{cases} x_1' = x_2 \\ x_2' = -x_1 \end{cases} \quad \Rightarrow \quad \boldsymbol{F}(x_1, x_2) = \langle x_2, -x_1 \rangle.$$



Example 5: Use the direction field of the predator prey system below to draw qualitative graphs of the solutions to that system,

$$R' = 5\,R - 2\,RF$$
$$F' = -F + \frac{1}{2}\,RF$$





Interactive Graph: Predator-Prey System.

Example 6. (Look at Home): Consider the predator prey system

$$x' = 2x - \frac{1}{5}x^2 - xy$$
$$y' = -5y + \frac{3}{2}xy.$$

The direction field of that system is given below, with one **solution curve** plotted in red.

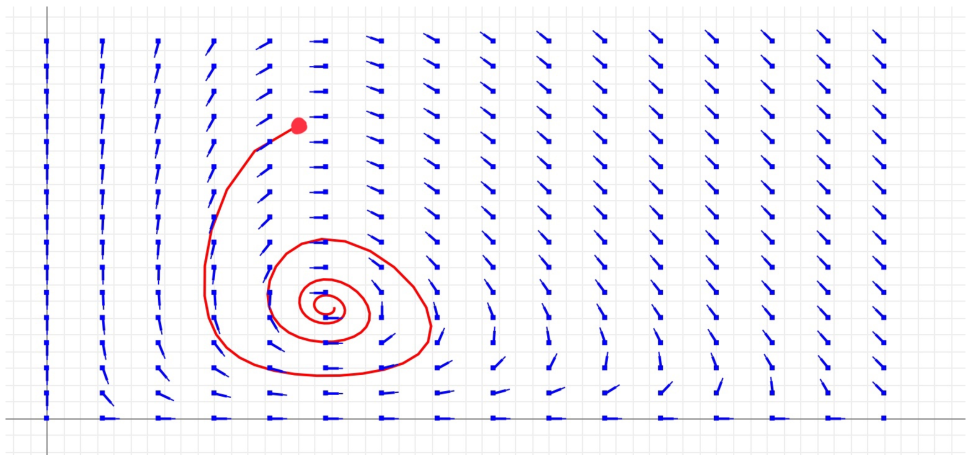


FIGURE 5. The horizontal x-axis represents the prey, the vertical y-axis represents the predator. The red dot highlights an initial condition. (The vectors point in the direction where the segments get thinner.)

Each component of the solution curve plotted in red above are plotted below as functions of time.

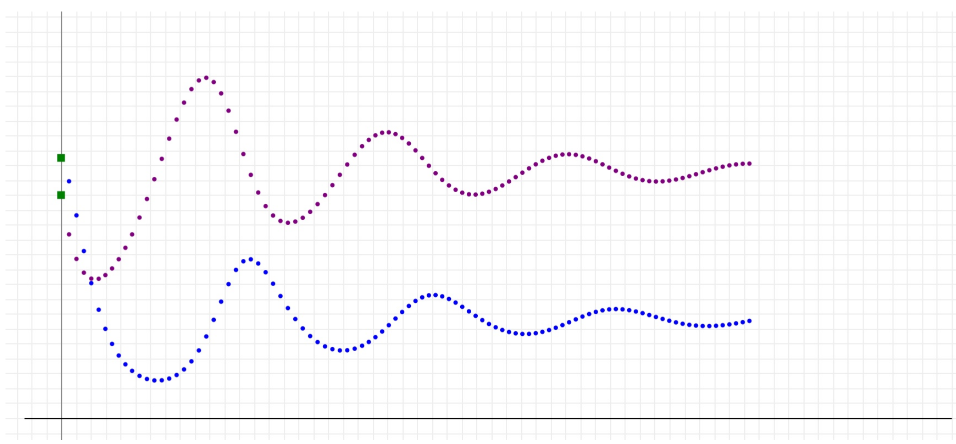


FIGURE 6. The horizontal t-axis represents time, and in the vertical axis we plot the the prey population (in purple) and the predator population (in blue).

Interactive Graph: Predator-Prey System with Limited Food.

5.1. Systems of Algebraic Linear Equations

Section Objective(s):

- The Row Picture.
- The Column Picture.
- The Matrix Picture.

Remarks:

- Before trying to solve systems of differential equations we need to know basic concepts of linear algebra.
- Solving linear algebraic equations by substitution is called the row picture. (One equation at a time.)
- Linear algebraic equation can be thought as a linear combination of vectors. This is the column picture.
- The concept of vector space comes from the row picture.
- Linear algebraic equation can be thought as a a matrix acting on vectors. This is the matrix picture.
- From the matrix picture we get the idea that matrices are functions on the space of vectors.

5.1.1. The Row Picture.

Remark: The field of linear algebra started when people tried to solve systems of linear algebraic equations.

Example 1: Find all solutions (x, y) of the 2×2 linear system

$$2x - y = 0$$
$$-x + 2y = 3.$$

Provide both a geometrical and an analytical solution.

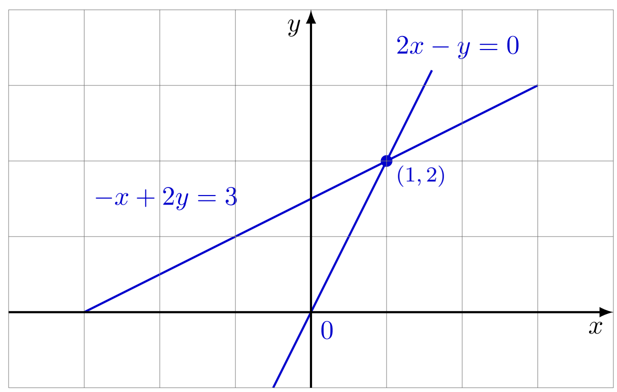
Solution:

$$y = 2x$$

$$-x + 4x = 3 \quad \Rightarrow \quad x = 1$$

$$y = 2.$$

Answer: $(x = 1, y = 2)$.



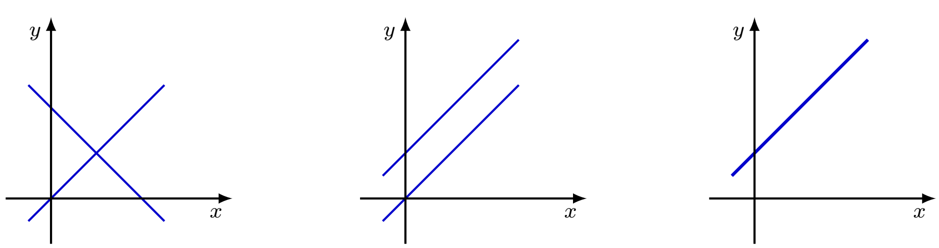
Theorem 1. Given a 2×2 linear system, only one of the following statements holds:

(i) There exists a unique solution;

(ii) There exist infinitely many solutions;

(iii) There exists no solution.

Proof:



5.1.2. The Column Picture.

Remark: The concept of a linear combination of vectors, and the idea of vector space come from the column picture.

Example 2: Write the system in Example 1 as a linear combination of column vectors,

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3. \end{aligned}$$

Solution:

We know that

$$\begin{aligned} (\mathbf{2})\, x + (\mathbf{-1})\, y &= \mathbf{0}, \\ (\mathbf{-1})\, x + (\mathbf{2})\, y &= \mathbf{3}. \end{aligned}$$

We introduce new objects, column vectors

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} y = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

We denote column vectors as follows,

$$\boldsymbol{a}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

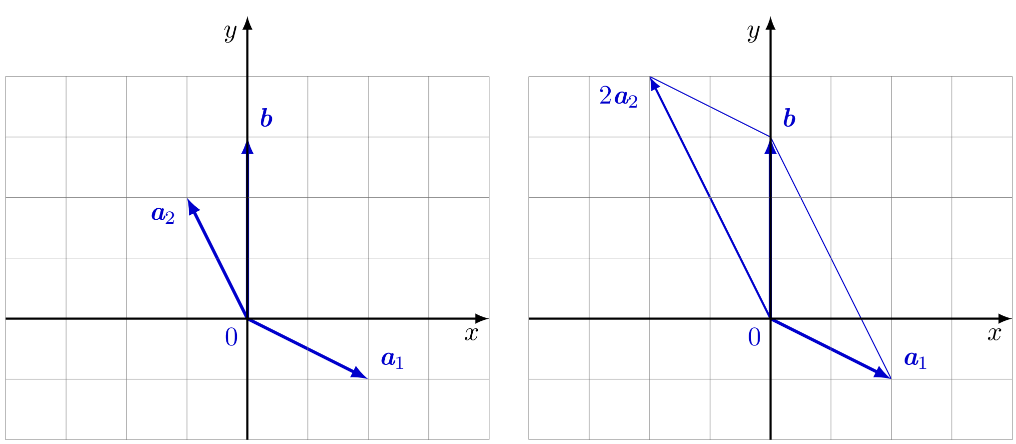
We can represent these vectors in the plane.

The solution is $(x = 1, y = 2)$. This defines the linear combination of column vectors,

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} 2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

So we define

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} 2 = \begin{bmatrix} (-1)2 \\ (2)2 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2-2 \\ -1+4 \end{bmatrix}.$$



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Remark: The example above is the motivation for the following definition.

Definition 1. The linear combination of the n-vectors

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

with the real numbers a and b, is defined as

$$a\boldsymbol{u} + b\boldsymbol{v} = a \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + b \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} au_1 + bv_1 \\ \vdots \\ au_n + bv_n \end{bmatrix}.$$

Remark: Theorem 1 can also be proven using linear combination of column vectors.

Theorem 1. Given a 2×2 linear system, only one of the following statements holds:

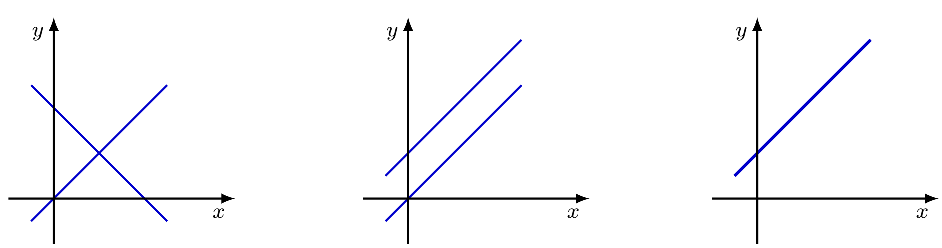
(i) There exists a unique solution;

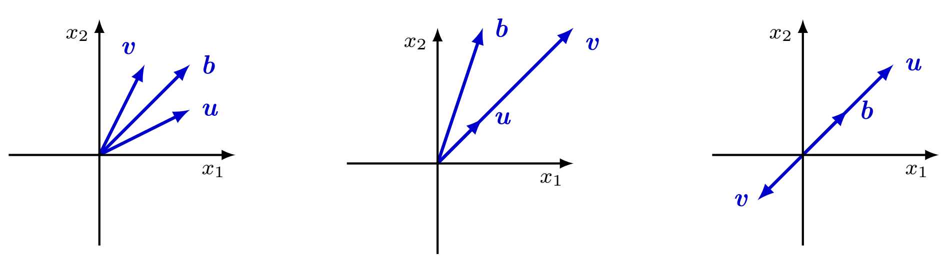
(ii) There exist infinitely many solutions;

(iii) There exists no solution.

Proof:

$$\left.\begin{array}{l} u_1 x + v_1 y = b_1 \\ u_2 x + v_2 y = b_2 \end{array}\right\} \quad \Leftrightarrow \quad \left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} x + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} y = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right..$$





5.1.3. The Matrix Picture.

Remark: The concept that a matrix is a function on vectors comes from the matrix picture.

Example 3: Write the system in Example 1 as a matrix acting on a column vector,

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3. \end{aligned}$$

Solution:

Rewrite the system as

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Define the matrix-vector product such that

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix},$$

Introduce the matrix notation

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Then, the linear system above is interpreted as:

$$A\boldsymbol{x} = \boldsymbol{b}.$$

Given A and $\boldsymbol{b}$ we need to find $\boldsymbol{x}$.

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Remark: The example above motivates the following definition.

Definition 2. An $m \times n$ matrix A is an array of numbers

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \qquad \begin{matrix} m \text{ rows}, \\ \\ n \text{ columns}, \end{matrix},$$

where $a_{ij} \in \mathbb{R}$ or $\mathbb{C}$, for $i = 1, \cdots, m$, $j = 1, \cdots, n$. A square matrix is an $n \times n$ matrix, and the diagonal coefficients in a square matrix are a_{ii}.

Remark: A matrix is a function that acts on a vector and the result is another vector.

Remark: We define linear combinations ________ and multiplications ________ of matrices.

- Addition of two matrices of the same size:

$$A+B=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

- Multiplication of a matrix A by a scalar c:

$$cA = c\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{bmatrix}$$

- Matrix multiplication is defined for matrices such that the numbers of columns in the first matrix matches ________ the numbers of rows in the second matrix.

$$\begin{array}{ccccc} A & \text{times} & B & \text{defines} & AB \\ m\times \boldsymbol{n} & & \boldsymbol{n}\times \ell & & m\times \ell \end{array}$$

$$AB=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11}+a_{12}b_{21}) & (a_{11}b_{12}+a_{12}b_{22}) \\ (a_{21}b_{11}+a_{22}b_{21}) & (a_{21}b_{12}+a_{22}b_{22}) \end{bmatrix}.$$

Example 4: Compute AB and BA, where $A=\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

Solution: We find that

$$AB=\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$

$$BA=\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Notice that in this case $AB \neq BA$. **The product is not commutative.**

Notice that $BA = 0$ but $A \neq 0$ and $B \neq 0$. ◁

5.2. Vector Spaces

Section Objective(s):

- Vector Spaces, Subspaces, and Spans.
- Linear (In)dependence.
- Basis and Dimension.

Remarks:

- The row picture of linear algebraic equations originates the idea of a vector space.
- A subspace is a smaller vector space inside a larger vector space.
- The span of a few vectors is the set of all linear combinations of these vectors.
- The span of a vectors creates a subspace .
- Vectors are linearly independent if none of them is linear combination of the others.
- A basis of V is the largest l.i. set in V.
- A basis of V is the smallest spanning set in V.
- The dimension of V is the number of basis vectors .
- The dimension of V measures how big is V.

5.2.1. Vector Spaces, Subspaces, and Spans.

Definition 1. The vector space $\mathbb{R}^n$, over $\mathbb{R}$, is the set of n-vectors with real components, together with the operation of linear combination,

$$\boldsymbol{u}+\boldsymbol{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}, \qquad a\boldsymbol{u} = a\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \\ \vdots \\ au_n \end{bmatrix}, \qquad a \in \mathbb{R}.$$

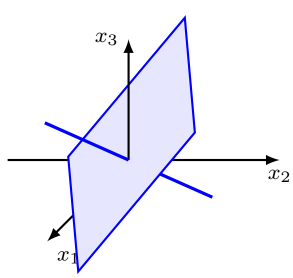
Remarks:

- The vector space $\mathbb{C}^n$, over $\mathbb{C}$, is the set of n vectors with complex components, together with the linear combination operation.
- We will use V to denote the vector space $\mathbb{R}^n$ or $\mathbb{C}^n$, and $\mathbb{F}$ to denote the field of scalars $\mathbb{R}$ of $\mathbb{C}$.

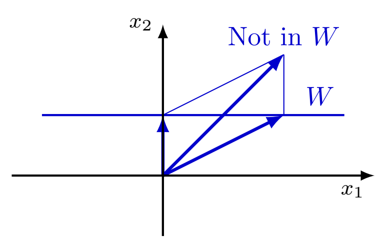
Definition 2. The subset $W \subseteq V$ of a vector space V over the field of scalars $\mathbb{F}$ is called a subspace of V iff for all $\boldsymbol{u}, \boldsymbol{v} \in W$ and all $a, b \in \mathbb{F}$,

$$a\boldsymbol{u} + b\boldsymbol{v} \in W$$

Example 1: Planes and lines through the **origin** are subspaces of $\mathbb{R}^3$.



A subspace **must** contain $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.



No circles, no nonlinear curves are subspaces.

Example 2: Which of the following sets W are subspaces of the vector space V?

(1) $V = \mathbb{R}^2$, $W = \left\{ \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ such that } u_2 = 0 \right\}$.

Yes, horizontal line containing $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(2) $V = \mathbb{R}^2$, $W = \left\{ \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ such that } u_2 = 1 \right\}$.

No, horizontal line not containing $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(3) $V = \mathbb{R}^2$, $W = \left\{ \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ such that } u_1 + u_2 = 0 \right\}$.

Yes, line slope -1 containing $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_2 \\ u_2 \end{bmatrix} = u_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(4) $V = \mathbb{R}^3$, $W = \left\{ \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ such that } u_2 = 2u_3 \right\}$.

Yes, plane of vectors $\begin{bmatrix} u_1 \\ 2u_3 \\ u_3 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

Remark: We finally introduce the definition of a span of a finite set of vectors.

Definition 3. The Span of a finite set $S = \{u_1, \dots, u_n\}$ in a vector space V over the field of scalars $\mathbb{F}$ is

$$\text{Span}(S) = \{\boldsymbol{u} \in V \text{ such that } \boldsymbol{u} = c_1\boldsymbol{u}_1 + \cdots + c_n\boldsymbol{u}_n, \text{ where } c_1, \dots, c_n \in \mathbb{F}\}.$$

Theorem 1. The Span(S) in a vector space V is a subspace of V.

Proof:

- The Span(S) contains all possible linear combinations of the elements in S.
- So Span(S) is a vector space.
- Span$(S) \subset V$, then the Span is a subspace.

Example 3: Give a geometric description of the following.

(1) Span$(\{\boldsymbol{v}_1, \boldsymbol{v}_2\})$ in $\mathbb{R}^3$, where $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$. Line

(2) Span$(\{\boldsymbol{v}_1, \boldsymbol{v}_2\})$ in $\mathbb{R}^3$, where $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$. Plane

5.2.2. Linear (In)dependence.

Definition 4. A finite set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ in a vector space is called linearly dependent iff there exists a set of scalars $\{c_1, \ldots, c_k\}$, **not all of them zero**, such that,

$$c_1 \boldsymbol{v}_1 + \cdots + c_k \boldsymbol{v}_k = 0.$$

The set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is called linearly independent iff the **only solution** of the equation above is

$$c_1 = 0, \ \ldots, \ c_k = 0.$$

Remarks:

- Linear dependence means a vector is l.c. of the others.
- Linear independence means no vector is l.c. of the others.

Example 4: Determine if the following sets are linearly independent and justify your claim.

(1) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$. Linear dependent.

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Leftrightarrow \quad 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(2) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \right\}$. Linear independent.

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} c_1 = 0 \\ c_2 + 3c_3 = 0 \\ 2c_2 = 0, \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0. \end{cases}$$

5.2.3. Basis and Dimension.

Definition 5. A set $S \subset V$ is a basis of a vector space $V = \{\mathbb{R}^n, \mathbb{C}^n\}$ iff

(1) S is linearly independent

and

(2) $\text{Span}(S) = V$.

Example 5: Determine if the following sets provide bases for the given vector space.

(1) $V = \mathbb{R}^3$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. No, S is too small.

(2) $V = \mathbb{R}^2$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$. No, the $\text{Span}(S)$ is the horizontal line containing $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(3) $V = \mathbb{R}^3$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \right\}$. Yes, S is l.i. and $\text{Span}(S) = \mathbb{R}^3$.

Theorem 2. The number of vectors ______________ in any basis of a vector space $V = \{\mathbb{R}^n, \mathbb{C}^n\}$ is the same as ______________ in any other basis of V.

Definition 7. The dimension __________ of a vector space $V = \{\mathbb{R}^n, \mathbb{C}^n\}$ is n, the number of vectors in any basis of V. ______________

Example 6: Give an example of two different bases of $\mathbb{R}^2$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

Example 7: Determine the dimension of the vector space given by

$$W = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \right\}$$

W has dimension 2.

5.3. Invertible Matrices. Eigenvalues and Eigenvectors

Section Objective(s):

- Invertible Matrices.
- Determinant of a Matrix.
- Eigenvalues and Eigenvectors of a Matrix.

Remarks:

- .

Example 1: Solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$ given below and find the matrix $\tilde{A}$ such that the solution can be written as $\boldsymbol{x} = \tilde{A}\boldsymbol{b}$.

$$A\,\boldsymbol{x} = \boldsymbol{b}, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{l} a\,x_1 + b\,x_2 = b_1 \\ c\,x_1 + d\,x_2 = b_2 \end{array}$$

$$\left.\begin{array}{r} d\,\big(a\,x_1 + b\,x_2 = b_1\big) \\ -b\,\big(c\,x_1 + d\,x_2 = b_2\big) \end{array}\right\} \quad \Rightarrow \quad (ad - bc)\,x_1 + (bd - bd)\,x_2 = d\,b_1 - b\,b_2$$

Introduce $\Delta = ad - bc$. **Assume that** $\Delta \neq 0$, then

$$x_1 = \frac{1}{\Delta}(d\,b_1 - b\,b_2).$$

A similar calculation gives

$$x_2 = \frac{1}{\Delta}(-c\,b_1 + a\,b_2).$$

This result can be written in matrix form as follows,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d\,b_1 - b\,b_2 \\ -c\,b_1 + a\,b_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \tilde{A}\,\boldsymbol{b},$$

so we get that

$$\tilde{A} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The matrix $\tilde{A}$ is the inverse of matrix A, and it is denoted as A^{-1}, that is,

$$A\boldsymbol{x} = \boldsymbol{b}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x} = A^{-1}\boldsymbol{b}, \quad A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \Delta = ad - bc \neq 0.$$

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Definition 1. A square matrix A is ***invertible*** iff there is a matrix A^{-1} so that

$$\big(A^{-1}\big)A = I_n \qquad A\big(A^{-1}\big) = I_n.$$

Example 2: Verify that the matrix and its inverse are given by

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

Solution:

We have to compute the products,

$$A\big(A^{-1}\big) = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \Rightarrow \quad A\big(A^{-1}\big) = I_2.$$

It is simple to check that the equation $\big(A^{-1}\big)A = I_2$ also holds.

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Theorem 1. Given a 2×2 matrix A, let Δ be the number

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \Delta = ad - bc.$$

Then, A is invertible iff $\Delta \neq 0$. Furthermore, if A is invertible, its inverse is

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Remarks:

(a) The number $\Delta = \det(A)$ is called the determinant of A.

(b) Δ determines whether A is invertible or not.

Example 3: Compute the inverse of matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, given in Example 8.2.15.

Solution: We first compute $\Delta = 6 - 4 = 4$. Since $\Delta \neq 0$, then A^{-1} exists and

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

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Example 4: Find a matrix X such that $AXB = I$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution:

Is A invertible? $\det(A) = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0$, so Yes.

Is B invertible? $\det(B) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0$, so Yes.

We then compute their inverses,

$$A^{-1} = \frac{1}{-5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}, \qquad B = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

We can now compute X,

$$AXB = I \quad \Rightarrow \quad A^{-1}(AXB)B^{-1} = A^{-1}IB^{-1} \quad \Rightarrow \quad X = A^{-1}B^{-1}.$$

Therefore,

$$X = \frac{1}{-5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = -\frac{1}{15} \begin{bmatrix} 5 & -7 \\ -5 & 4 \end{bmatrix}$$

so we obtain

$$X = \begin{bmatrix} -\frac{1}{3} & \frac{7}{15} \\ \frac{1}{3} & -\frac{4}{15} \end{bmatrix}.$$

◁

Definition 2. The ***determinant of a*** 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \underline{a_{11}a_{22} - a_{12}a_{21} \qquad\qquad\qquad\qquad}.$$

Definition 3. The ***determinant of a*** 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example 5: Compute the determinant of the 3×3 matrix,

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution:

$$\begin{aligned} \begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} &= (1) \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ &= (1-2) - 3\,(2-3) - (4-3) \\ &= -1 + 3 - 1 \\ &= 1. \end{aligned}$$

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Exercise: Show that the determinant of an upper triangular matrix (one all of whose entries below the main diagonal are zero) is the product of the diagonal coefficients. How about a lower triangular matrix?

5.3.1. Eigenvalues and Eigenvectors of a Matrix.

Definition 4. A number λ and a nonzero n-vector $\boldsymbol{v}$ are an ***eigenvalue*** and ***eigenvector*** (eigenpair) of a square matrix A iff they satisfy the equation

$$A\boldsymbol{v} = \lambda \boldsymbol{v} \; .$$

Remarks:

(a) An eigenvector $\boldsymbol{v}$ determines a particular direction in the space that remains invariant under the action of the matrix A.

(b) That is, if $\boldsymbol{v}$ is an eigenvector, so is $a\boldsymbol{v}$ for $a \in \mathbb{R}$.

$$A(a\boldsymbol{v}) = a\,A\boldsymbol{v} = a\lambda\boldsymbol{v} = \lambda(a\boldsymbol{v}) \; .$$

Example 6: Verify that the pair λ_1, $\boldsymbol{v}_1$ and the pair λ_2, $\boldsymbol{v}_2$ are eigenvalue and eigenvector pairs of matrix A given below,

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad \begin{cases} \lambda_1 = 4 & \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \lambda_2 = -2 & \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \end{cases}$$

Solution: We just must verify the definition of eigenvalue and eigenvector given above. We start with the first pair,

$$A\boldsymbol{v}_1 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \boldsymbol{v}_1 \quad \Rightarrow \quad A\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1.$$

A similar calculation for the second pair implies,

$$A\boldsymbol{v}_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_2 \boldsymbol{v}_2 \quad \Rightarrow \quad A\boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_2.$$

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Remark: How do we find the eigenvalues and eigenvectors of a square matrix?

Theorem 2. (Eigenvalues-Eigenvectors)

(a) All the eigenvalues λ of an $n \times n$ matrix A are the solutions of

$$\underline{\det(A - \lambda I) = 0}\ .$$

(b) Given an eigenvalue λ of an $n \times n$ matrix A, the corresponding eigenvectors $\boldsymbol{v}$ are the nonzero solutions to the homogeneous linear system

$$\underline{(A - \lambda I)\boldsymbol{v} = \boldsymbol{0}}\ .$$

Remark: An eigenvalue λ is a number such that $A - \lambda I$ is <u>not invertible</u>.

Example 7: Find the eigenvalues λ and eigenvectors $\boldsymbol{v}$ of the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: We first find the eigenvalues as the solutions of the Eq. (**??**). Compute

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{bmatrix}.$$

Then we compute its determinant,

$$0 = \det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9 \quad \Rightarrow \quad \begin{cases} \lambda_+ = 4, \\ \lambda_- = -2. \end{cases}$$

We have obtained two eigenvalues, so now we introduce $\lambda_+ = 4$ into Eq. (**??**), that is,

$$A - 4I = \begin{bmatrix} 1-4 & 3 \\ 3 & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}.$$

Then we solve for $\boldsymbol{v}^+$ the equation

$$(A - 4I)\boldsymbol{v}^+ = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution can be found using Gauss elimination operations, as follows,

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^+ = v_2^+, \\ v_2^+ \quad \text{free}. \end{cases}$$

All solutions to the equation above are then given by

$$\boldsymbol{v}^{+} = \begin{bmatrix} v_2^{+} \\ v_2^{+} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2^{+} \quad \Rightarrow \quad \boldsymbol{v}^{+} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where we have chosen $v_2^{+} = 1$. A similar calculation provides the eigenvector $\boldsymbol{v}^{-}$ associated with the eigenvalue $\lambda_{-} = -2$, that is, first compute the matrix

$$A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

then we solve for $\boldsymbol{v}^{-}$ the equation

$$(A + 2I)\boldsymbol{v}^{-} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1^{-} \\ v_2^{-} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution can be found using Gauss elimination operations, as follows,

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{-} = -v_2^{-}, \\ v_2^{-} \quad \text{free.} \end{cases}$$

All solutions to the equation above are then given by

$$\boldsymbol{v}^{-} = \begin{bmatrix} -v_2^{-} \\ v_2^{-} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2^{-} \quad \Rightarrow \quad \boldsymbol{v}^{-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where we have chosen $v_2^{-} = 1$. We therefore conclude that the eigenvalues and eigenvectors of the matrix A above are given by

$$\lambda_{+} = 4, \quad \boldsymbol{v}^{+} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_{-} = -2, \quad \boldsymbol{v}^{-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

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5.4. Eigenvalues and Eigenvectors.

Section Objective(s):

- Eigenvalues and Eigenvectors.
- Computing Eigenpairs.

Remarks:

- A matrix acting on a vector usually changes ____________ the direction of the vector.
- An eigenvector of a matrix A determines a particular direction ________________ in space that is invariant ______________ under the action of A.
- The eigenvectors of the coefficient matrix ___________________________ of a linear differential ________________ system will play an important role to find solutions to the system.

5.4.1. Eigenvalues and Eigenvectors.

Definition 1. A number λ and a nonzero n-vector $\boldsymbol{v}$ are an eigenvalue and eigenvector also called eigenpair, of a square matrix A iff they satisfy the equation

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

Remark: The length of an eigenvector is not important, because if $\boldsymbol{v}$ is an eigenvector, so is $(a\boldsymbol{v})$ for $a \in \mathbb{R}$.

$$A(a\boldsymbol{v}) = a\,A\boldsymbol{v} = a\lambda\boldsymbol{v} = \lambda(a\boldsymbol{v}).$$

Example 1: Verify that the pair λ_1, $\boldsymbol{v}_1$ and the pair λ_2, $\boldsymbol{v}_2$ are eigenpairs of matrix A,

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad \begin{cases} \lambda_1 = -1 & \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\ \lambda_2 = -2 & \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{cases}$$

Solution: We just must verify the definition of eigenvalue and eigenvector given above. We start with the first pair,

$$A\boldsymbol{v}_1 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda_1 \boldsymbol{v}_1 \quad \Rightarrow \quad A\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1.$$

A similar calculation for the second pair implies,

$$A\boldsymbol{v}_2 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \lambda_2 \boldsymbol{v}_2 \quad \Rightarrow \quad A\boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_2.$$

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Interactive Graph: A Geometrical Meaning of Eigenpairs.

5.4.2. Computing Eigenpairs.

Remarks:

- The eigenpairs equation can be written as

$$(A - \lambda I)\,\boldsymbol{v} = \mathbf{0}, \qquad \boldsymbol{v} \neq \mathbf{0}.$$

- This equation says that the matrix $(A - \lambda I)$ is not invertible.
- There is a way to determine whether this matrix is invertible,

$$\det(A - \lambda I) = 0.$$

Theorem 1. (Eigenvalues-Eigenvectors)

(a) All the eigenvalues λ of an $n \times n$ matrix A are the solutions of

$$\det(A - \lambda I) = 0.$$

(b) Given an eigenvalue λ of an $n \times n$ matrix A, the corresponding eigenvectors $\boldsymbol{v}$ are the nonzero solutions to the homogeneous linear system

$$(A - \lambda I)\boldsymbol{v} = \mathbf{0}.$$

Remarks:

- We look for numbers λ such that the matrix $(A - \lambda I)$ is not invertible.
- Given an $n \times n$ matrix A, the function $p(\lambda) = \det(A - \lambda I)$ is a polynomial degree n.
- This polynomial $p(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial of matrix A.

Example 2: Find the eigenvalues λ and eigenvectors $\boldsymbol{v}$ of the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution:

We first find the eigenvalues as the solutions of the equation $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{bmatrix}.$$

Then we compute its determinant,

$$0 = \det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9 \quad \Rightarrow \quad \begin{cases} \lambda_+ = 4, \\ \lambda_- = -2. \end{cases}$$

We have obtained two eigenvalues, so now we introduce $\lambda_+ = 4$ into Eq. (**??**), that is,

$$A - 4I = \begin{bmatrix} 1-4 & 3 \\ 3 & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}.$$

Then we solve for $\boldsymbol{v}^+$ the equation

$$(A - 4I)\boldsymbol{v}^+ = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution can be found using Gauss elimination operations, as follows,

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^+ = v_2^+, \\ v_2^+ \quad \text{free}. \end{cases}$$

All solutions to the equation above are then given by

$$\boldsymbol{v}^+ = \begin{bmatrix} v_2^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2^+ \quad \Rightarrow \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where we have chosen $v_2^+ = 1$. A similar calculation provides the eigenvector $\boldsymbol{v}^-$ associated with the eigenvalue $\lambda_- = -2$, that is, first compute the matrix

$$A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

then we solve for $\boldsymbol{v}^{-}$ the equation

$$(A+2I)\boldsymbol{v}^{-} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1^{-} \\ v_2^{-} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution can be found using Gauss elimination operations, as follows,

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{-} = -v_2^{-}, \\ v_2^{-} \quad \text{free.} \end{cases}$$

All solutions to the equation above are then given by

$$\boldsymbol{v}^{-} = \begin{bmatrix} -v_2^{-} \\ v_2^{-} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2^{-} \quad \Rightarrow \quad \boldsymbol{v}^{-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where we have chosen $v_2^{-} = 1$. We therefore conclude that the eigenvalues and eigenvectors of the matrix A above are given by

$$\lambda_{+} = 4, \quad \boldsymbol{v}^{+} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_{-} = -2, \quad \boldsymbol{v}^{-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$\triangleleft$

Example 3: Find the eigenvalues λ and eigenvectors $\boldsymbol{v}$ of the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

Solution:

We first find the eigenvalues as the solutions of the equation $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (2-\lambda) & 1 \\ 0 & (2-\lambda) \end{bmatrix}.$$

Then we compute its determinant,

$$0 = \det(A - \lambda I) = \begin{vmatrix} (2-\lambda) & 1 \\ 0 & (2-\lambda) \end{vmatrix} = (\lambda - 2)^2 - 0 \quad \Rightarrow \quad \lambda_0 = 2.$$

We have obtained only eigenvalue. Now we introduce it into the equation voe $\boldsymbol{v}$,

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then we solve for $\boldsymbol{v}$ the equation

$$(A - 2I)\boldsymbol{v} = \mathbf{0} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_2 = 0.$$

All solutions to the equation above are then given by

$$\boldsymbol{v} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_1 \quad \Rightarrow \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where we have chosen $v_1^+ = 1$. So we have only one eigenpair,

$$\lambda = 2, \qquad \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

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5.5. Diagonalizable Matrices

Section Objective(s):

- Diagonal Matrices.
- Diagonalizable Matrix.

Remarks:

- Diagonal matrices are simple to work with, but they do not appear so often in physical applications.
- General matrices are difficult to work with, since the matrix product is complicated and not commutative.
- Diagonalizable matrices are an intermediate case:
  - They are general enough to often appear in physical applications.
  - The are simple enough to work with.
  - Functions of diagonalizable matrices are simple to compute.

5.5.1. Diagonal Matrices.

Definition 1. An $n \times n$ matrix A is diagonal iff

$$A = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}.$$

Remarks:

- Notation: $\begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix} = \operatorname{diag}\left[a_{11}, \cdots, a_{nn}\right].$
- Matrix operations are simple with diagonal matrices.

Example 1: Given $A = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$, compute A^2, A^3, and A^n for a general natural number n.

Solution:

$$A^2 = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & 7^2 \end{bmatrix},$$

$$A^3 = A^2 A = \begin{bmatrix} 2^2 & 0 \\ 0 & 7^2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & 7^2 \end{bmatrix}$$

By induction, using $A^n = A^{(n-1)} A$, one gets

$$A^n = \begin{bmatrix} 2^n & 0 \\ 0 & 7^n \end{bmatrix}.$$

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Remarks: Consider a diagonal matrix $D = \operatorname{diag}\left[a_{11}, \cdots, a_{nn}\right]$:

- Then $D^n = \operatorname{diag}\left[a_{11}^n, \cdots, a_{nn}^n\right]$.
- The eigenvalues of a D are $a_{11}, \cdots, a_{nn}$.
- The corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \boldsymbol{v}_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}, \quad \text{since for example} \quad A = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} = a_{11} \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}.$$

5.5.2. Diagonalizable Matrices.

Remarks:

- Diagonal matrices do not appear often in physical applications.
- But diagonalizable matrices are very common in physical aplications.

Definition 2. A square matrix A is diagonalizable iff there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

Remark: $A = PDP^{-1}$ is equivalent to $P^{-1}AP = D$.

Example 2: Show that the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Solution:

$$A = PDP^{-1} \quad \Rightarrow \quad P^{-1}AP = D, \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} = D$$

Remark: Notice, $P = [\boldsymbol{v}_1, \boldsymbol{v}_2]$, $D = \mathrm{diag}[\lambda_1, \lambda_2]$, with $\boldsymbol{v}_1$, λ_1 and $\boldsymbol{v}_2$, λ_2 eigenpairs of A,

$$\boldsymbol{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \lambda_1 = 4, \quad \lambda_2 = -2.$$

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Example 3: If A is a 2×2 with eigenpairs λ_1, $\boldsymbol{v}_1$ and λ_2, $\boldsymbol{v}_2$, then show that

$$AP = PD, \qquad \text{where} \qquad P = \big[\boldsymbol{v}_1, \boldsymbol{v}_2\big], \qquad D = \operatorname{diag}\big[\lambda_1, \lambda_2\big].$$

Solution:

$$AP = A\left[\boldsymbol{v}_1, \boldsymbol{v}_2\right] = \left[A\boldsymbol{v}_1, A\boldsymbol{v}_2\right] = \left[\lambda_1\boldsymbol{v}_1, \lambda_2\boldsymbol{v}_2\right] = \left[\boldsymbol{v}_1, \boldsymbol{v}_2\right]\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix} = PD.$$

Since

$$AP = PD \quad \Rightarrow \quad A = PDP^{-1}.$$

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Remark: The next result says that this result and its converse are true for $n \times n$ matrices.

Theorem 1. An $n \times n$ matrix A is diagonalizable iff A has n eigenvectors linearly independent.

Furthermore, if $\lambda_i, \boldsymbol{v}_i$, for $i = 1, \cdots, n$, are eigenpairs of A, then $A = PDP^{-1}$, where

$$P = \big[\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\big], \qquad D = \mathrm{diag}\big[\lambda_1, \cdots, \lambda_n\big].$$

Example 4: Is the matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. diagonalizable?

Solution:

- $0 = \det(A - \lambda I) = \begin{vmatrix} (3-\lambda) & 0 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = (3-\lambda)^2(1-\lambda)$
- So $\lambda_1 = 1$, $\lambda_2 = 3$.
- For $\lambda_1 = 1$, $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $\begin{cases} 2v_1 = -v_3 \\ v_2 = -v_3 \\ v_3 \text{ free} \end{cases} \Rightarrow \quad \boldsymbol{v}_1 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$.
- For $\lambda_2 = 3$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $\begin{cases} v_1 \text{ free} \\ v_2 \text{ free} \Rightarrow \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \\ v_3 = 0 \end{cases}$
- Then, A has three eigenvectors linearly independent, so A is diagonalizable and

$$A = PDP^{-1}, \quad \text{where} \quad D = \mathrm{diag}\big[1, 3, 3\big], \quad P = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

◁

Remark: Matrix P is not unique, since the eigenvectors are not unique.

Another choice is $A = \tilde{P}\tilde{D}\tilde{P}^{-1}$ with $\tilde{P} = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 0 & 5 \\ -4 & 0 & 0 \end{bmatrix}$ $\tilde{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Example 5: Is the matrix $B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizable?

Solution:

- $0 = \det(B - \lambda I) = \begin{vmatrix} (3-\lambda) & 1 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = (3-\lambda)^2(1-\lambda)$
- So $\lambda_1 = 1$, $\lambda_2 = 3$.
- For $\lambda_1 = 1$, $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $\begin{cases} 2v_1 + v_2 = -v_3 \\ v_2 = -v_3 \\ v_3 \text{ free} \end{cases} \Rightarrow \boldsymbol{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.
- For $\lambda_2 = 3$, $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $\begin{cases} v_1 + v_3 = 0 \\ v_2 \text{ free} \\ v_3 = 0 \end{cases} \Rightarrow \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.
- Then, B has only two eigenvectors linearly independent, so B is not diagonalizable.

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5.6. The Matrix Exponential

Section Objective(s):

- The Exponential of a Matrix.
  - Diagonal Matrices.
  - Diagonalizable Matrix.
- Properties of the Matrix Exponential.

Remarks:

- We know how to compute linear combinations ______ of matrices.
- We know how to compute multiplication ______ of matrices
- With these operation it is possible to define functions ______ of matrices.
- We define functions ______ of matrices using power series. ______

5.6.1. The Exponential of a Matrix.

Review: Recall the definition of the exponential of real numbers.

- $f(x) = e^x$ is defined as:
  - For n natural number, $e^n = e \cdots e$, for n-times.
  - Then, $e^0 = 1$, and for negative integers $-n$
$$e^{-n} = \frac{1}{e^n}.$$
  - Then, for rational numbers, m/n, with m, n integers,
$$e^{\frac{m}{n}} = \sqrt[n]{e^m}.$$
  - Then, for irrational numbers x, is done by a limit,
$$e^x = \lim_{\frac{m}{n} \to x} e^{\frac{m}{n}}.$$

  It is not clear how to extend this definition to matrices.
- The exponential is the inverse of the natural log:
$$e^x = y \quad \Leftrightarrow \quad \ln(y) = x,$$
  and $\ln(y)$, is
$$\ln(x) = \int_1^x \frac{1}{y}\, dy, \qquad x \in (0, \infty).$$
  It is not clear how to extend this definition to matrices.
- The exponential function can be defined also by its Taylor series,
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$
  This series expression can be generalized square matrices.

Definition 1. The exponential of a square matrix A is
$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots.$$

Remark: It can be shown that the infinite sum above converges for all square matrices.

5.6.2. The Exponential of a Matrix: Diagonal Matrices.

Example 1: Compute e^A, where $A = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$.

Solution:

We start with the definition of the exponential

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}^n .$$

But,

$$\begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}^2 = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & 7^2 \end{bmatrix}.$$

It is simple to see that, since the matrix A is diagonal,

$$\begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}^n = \begin{bmatrix} 2^n & 0 \\ 0 & 7^n \end{bmatrix}.$$

Therefore,

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} 2^n & 0 \\ 0 & 7^n \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} \frac{2^n}{n!} & 0 \\ 0 & \frac{7^n}{n!} \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{2^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{7^n}{n!} \end{bmatrix}.$$

Since $\displaystyle\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a$, for $a = 2, 7$, we obtain that

$$e^{\begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}} = \begin{bmatrix} e^2 & 0 \\ 0 & e^7 \end{bmatrix}.$$

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Theorem 1. If $D = \operatorname{diag}\left[d_1, \cdots, d_n\right]$, then

$$e^{\operatorname{diag}\left[d_1, \cdots, d_n\right]} = \operatorname{diag}\left[e^{d_1}, \cdots, e^{d_n}\right].$$

5.6.3. The Exponential of a Matrix: Diagonalizable Matrices.

Remarks:

- The exponential of a diagonalizable matrix is also simple to compute.
- We start computing powers of a diagonalizable matrix.

Theorem 2. If A is diagonalizable, with

$$A = PDP^{-1} = P\,\mathrm{diag}\big[a_{11}, \cdots, a_{nn}\big]\,P^{-1},$$

then

$$A^2 = PD^2P^{-1} = P\,\mathrm{diag}\left[(a_{11})^2, \cdots, (a_{nn})^2\right]\,P^{-1},$$
$$A^n = PD^nP^{-1} = P\,\mathrm{diag}\left[(a_{11})^n, \cdots, (a_{nn})^n\right]\,P^{-1}.$$

Proof of Theorem 2:

First the case A^2,

$$\begin{aligned} A^2 &= A\,A \\ &= (PDP^{-1})(PDP^{-1}) \\ &= PD^2P^{-1} \end{aligned}$$

Then induction. Assume that $A^{n-1} = PD^{n-1}P^{-1}$, then

$$\begin{aligned} A^n &= A^{n-1}\,A \\ &= (PD^{n-1}P^{-1})(PDP^{-1}) \\ &= PD^nP^{-1} \end{aligned}$$

□

Theorem 3. The exponential of a diagonalizable matrix A, with $A = PDP^{-1}$ and D diagonal, is

$$e^A = Pe^D P^{-1}.$$

Proof of Theorem 3: Recall that

$$A^n = PD^n P^{-1}.$$

We then compute the exponential of A as follows,

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^n = \sum_{k=0}^{\infty} \frac{1}{k!} (PDP^{-1})^n = \sum_{k=0}^{\infty} \frac{1}{k!} (PD^n P^{-1}),$$

On the far right we can take common factor P on the left and P^{-1} on the right,

$$e^A = P\Big(\sum_{k=0}^{\infty} \frac{1}{k!} D^n\Big) P^{-1}.$$

The sum in between parenthesis is e^D,

$$e^A = Pe^D P^{-1}.$$

This establishes the Theorem.

$\square$

Example 2: Compute e^{At}, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ and $t \in \mathbb{R}$.

Solution: To compute e^{At} we need the decomposition $A = PDP^{-1}$, which in turns implies that $At = P(Dt)P^{-1}$. Matrices P and D are constructed with the eigenvectors and eigenvalues of matrix A. We computed them in the previous examples.

$$\lambda_1 = 4, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -2, \quad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Introduce P and D as follows,

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

Then, the exponential function is given by

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Usually one leaves the function in this form. If we multiply the three matrices out we get

$$e^{At} = \frac{1}{2} \begin{bmatrix} (e^{4t} + e^{-2t}) & (e^{4t} - e^{-2t}) \\ (e^{4t} - e^{-2t}) & (e^{4t} + e^{-2t}) \end{bmatrix}.$$

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5.6.4. Properties of the Matrix Exponential.

Remark: We now summarize the main properties of the matrix exponential.

Theorem 4. If A is an $n \times n$ matrix and s, t are real numbers, then

- Group property
$$e^{As}\, e^{At} = e^{A(s+t)}.$$
- Inverse exponential
$$\big(e^{A}\big)^{-1} = e^{-A}.$$
- Derivative of the exponential,
$$\frac{d}{dt} e^{At} = A\, e^{At} = e^{At}\, A.$$
- If A, B are $n \times n$ matrices such that $AB = BA$, then
$$e^{A+B} = e^{A}\, e^{B}.$$

6.1. Two-Dimensional Linear Systems

Section Objective(s):

- 2×2 Linear Differential Systems.
- Diagonalizable Systems.
  - Real Distinct Eigenvalues.
  - Complex Eigenvalues.
  - Repeated Eigenvalues.
- Non-Diagonalizable Systems.
  - Repeated Eigenvalues.

Remarks:

- We introduce 2×2 systems of linear differential equations.
- We focus on homogeneous systems with constant coefficients.
- If the homogeneous linear differential system is diagonalizable, then we have a formula for all the solutions.
- If the homogeneous linear differential system is not diagonalizable, then the formula above give only half the solutions.
- The other half of the solutions can be found generalizing ideas from second order scalar equations with repeated roots of their characteristic polynomial.

6.1.1. 2×2 Linear Differential Systems.

Definition 1. A 2×2 ***first order linear differential system*** is the equation

$$\boldsymbol{x}'(t) = A(t)\,\boldsymbol{x}(t) + \boldsymbol{b}(t),$$

where the coefficient matrix A, the source vector $\boldsymbol{b}$, and the unknown vector $\boldsymbol{x}$ are

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}, \qquad \boldsymbol{b}(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}, \qquad \boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

The system above is called:

- homogeneous iff $\boldsymbol{b} = \boldsymbol{0}$,
- of constant coefficients iff A is constant,
- diagonalizable iff A is diagonalizable.

Remarks:

- In this class we focus on homogeneous systems with constant coefficients.
- Diagonal systems are very simple to solve.

Example 1: Find functions x_1, x_2 solutions of the first order, 2×2, constant coefficients, homogeneous differential system

$$x_1' = 3x_1,$$
$$x_2' = 2x_2.$$

Solution: In this case, the system is decoupled, so we are just solving 2 (independent) scalar equations. Recall that $x_1(t) = c_1 e^{3t}$ and $x_2(t) = c_2 e^{2t}$. In vector notation we get

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1\, e^{3t} \\ c_2\, e^{2t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}$$

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6.1.2. Diagonalizable Systems: Real Eigenvalues.

Example 2: Now, we consider a system where the equations are **coupled**. Find functions x_1, x_2 solutions of the follwoing system of ODEs

$$x_1' = x_1 + 3x_2,$$
$$x_2' = 3x_1 + x_2.$$

Solution: We saw that solving a decoupled system (diagonal matrix) is easy. If we have a diagonalizable matrix, $A = PDP^{-1}$, i.e., $D = P^{-1}AP$. Multiply the differential equation $\boldsymbol{x}' = A\boldsymbol{x}$ by P^{-1},

$$(P^{-1}\boldsymbol{x})' = (P^{-1}AP)(P^{-1}\boldsymbol{x}),$$

so introduce $\boldsymbol{y} = P^{-1}\boldsymbol{x}$, and the equation of $\boldsymbol{y}$ is then

$$\boldsymbol{y}' = D\boldsymbol{y}.$$

To find P and D for the given matrix, find the eigenpairs of A. The solution is

$$\lambda_1 = 4, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_- = -2, \quad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Therefore, matrix A is diagonalizable with

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}, \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

That is,

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1 + x_2 \\ -x_1 + x_2 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} y_1 = \dfrac{1}{2}(x_1 + x_2) \\ y_2 = \dfrac{1}{2}(-x_1 + x_2) \end{cases}.$$

The differential equation for $\boldsymbol{y}$ is $\boldsymbol{y}' = D\,\boldsymbol{y}$, hence

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} y_1' = 4y_1 \\ y_2' = -2y_2 \end{cases} \quad \Rightarrow \quad \begin{cases} y_1 = c_1\, e^{4t} \\ y_2 = c_2\, e^{-2t} \end{cases}.$$

We now transform back to $\boldsymbol{x} = P\boldsymbol{y}$,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1\, e^{4t} \\ c_2\, e^{-2t} \end{bmatrix} = \begin{bmatrix} c_1\, e^{4t} - c_2\, e^{-2t} \\ c_1\, e^{4t} + c_2\, e^{-2t} \end{bmatrix}$$

that is,

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

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Theorem 1. (Homogeneous Diagonalizable Systems) If an $n \times n$ constant matrix A is diagonalizable, with eigenpairs

$$(\lambda_1, \boldsymbol{v}_1), \cdots, (\lambda_n, \boldsymbol{v}_n),$$

then the general solution of $\boldsymbol{x}' = A\,\boldsymbol{x}$ is

$$\boldsymbol{x}(t) = c_1\, e^{\lambda_1 t}\, \boldsymbol{v}_1 + \cdots + c_n\, e^{\lambda_n t}\, \boldsymbol{v}_n.$$

Remark: Each function $\boldsymbol{x}_k(t) = e^{\lambda_k t}\, \boldsymbol{v}_k$ is solution of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$, because

$$\boldsymbol{x}_k' = \lambda_k\, e^{\lambda_k t}\, \boldsymbol{v}_k,$$

$$A\boldsymbol{x}_k = \big(A\, \boldsymbol{v}_k\big)\, e^{\lambda_k t} = \big(\lambda_k\, \boldsymbol{v}_k\big)\, e^{\lambda_k t} = \lambda_k\, e^{\lambda_k t}\, \boldsymbol{v}_k.$$

Example 3: Use the theorem above to find the general solution of the IVP

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} 3 & -2 \\ 10 & -6 \end{bmatrix}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Solution: Find the eigenpairs of A. The solution is

$$\lambda_+ = -1, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \lambda_- = -2, \quad \boldsymbol{v}^- = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

So the general solution is

$$\boldsymbol{x}(t) = c_+\, e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_-\, e^{-2t} \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Now we find the coefficients c_+ and c_- that satisfy the initial condition

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \boldsymbol{x}(0) = c_+ \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_- \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The inverse of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{5-4} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}^{-1} \quad \Rightarrow \quad \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We conclude that $c_+ = -1$ and $c_- = 1$, hence

$$\boldsymbol{x}(t) = -e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-2t} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \Leftrightarrow \quad \boldsymbol{x}(t) = \begin{bmatrix} -e^{-t} + 2\, e^{-2t} \\ -2\, e^{-t} + 5\, e^{-2t} \end{bmatrix}.$$

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6.1.3. Diagonalizable Systems: Complex Eigenvalues.

Remarks:

- A real matrix can have complex eigenvalues.
- But in this case, the eigenpairs come in conjugate pairs, $\lambda_- = \overline{\lambda}_+$, and $\boldsymbol{v}_- = \overline{\boldsymbol{v}}_+$.

Theorem 2. (Complex and Real Solutions) If a 2×2 matrix A has eigenpairs

$$\lambda_\pm = \alpha \pm i\beta, \qquad \boldsymbol{v}_\pm = \boldsymbol{a} \pm i\,\boldsymbol{b},$$

where α, β, $\boldsymbol{a}$, and $\boldsymbol{b}$ real, then the equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ has fundamental solutions

$$\boldsymbol{x}_+(t) = e^{\lambda_+ t}\,\boldsymbol{v}_+, \quad \boldsymbol{x}_-(t) = e^{\lambda_- t}\,\boldsymbol{v}_-,$$

but it also has ***real-valued*** fundamental solutions

$$\boldsymbol{x}_1(t) = \big(\boldsymbol{a}\,\cos(\beta t) - \boldsymbol{b}\,\sin(\beta t)\big)\,e^{\alpha t},$$

$$\boldsymbol{x}_2(t) = \big(\boldsymbol{a}\,\sin(\beta t) + \boldsymbol{b}\,\cos(\beta t)\big)\,e^{\alpha t}.$$

Proof of Theorem 2: We know that the solutions $\boldsymbol{x}_\pm$ are linearly independent. Now,

$$\begin{aligned}
\boldsymbol{x}_\pm &= (\boldsymbol{a} \pm i\boldsymbol{b})\,e^{(\alpha \pm i\beta)t} \\
&= e^{\alpha t}(\boldsymbol{a} \pm i\boldsymbol{b})\,e^{\pm i\beta t} \\
&= e^{\alpha t}(\boldsymbol{a} \pm i\boldsymbol{b})\,\big(\cos(\beta t) \pm i\sin(\beta t)\big) \\
&= e^{\alpha t}\big(\boldsymbol{a}\,\cos(\beta t) - \boldsymbol{b}\,\sin(\beta t)\big) \pm i e^{\alpha t}\big(\boldsymbol{a}\,\sin(\beta t) + \boldsymbol{b}\,\cos(\beta t)\big).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\boldsymbol{x}_+ &= e^{\alpha t}\big(\boldsymbol{a}\,\cos(\beta t) - \boldsymbol{b}\,\sin(\beta t)\big) + i e^{\alpha t}\big(\boldsymbol{a}\,\sin(\beta t) + \boldsymbol{b}\,\cos(\beta t)\big) \\
\boldsymbol{x}_- &= e^{\alpha t}\big(\boldsymbol{a}\,\cos(\beta t) - \boldsymbol{b}\,\sin(\beta t)\big) - i e^{\alpha t}\big(\boldsymbol{a}\,\sin(\beta t) + \boldsymbol{b}\,\cos(\beta t)\big).
\end{aligned}$$

Since the differential equation $\boldsymbol{x}' = A\boldsymbol{x}$ is linear, the functions below are also solutions,

$$\begin{aligned}
\boldsymbol{x}_1 &= \frac{1}{2}\big(\boldsymbol{x}^+ + \boldsymbol{x}^-\big) = \big(\boldsymbol{a}\,\cos(\beta t) - \boldsymbol{b}\,\sin(\beta t)\big)\,e^{\alpha t}, \\
\boldsymbol{x}_2 &= \frac{1}{2i}\big(\boldsymbol{x}^+ - \boldsymbol{x}^-\big) = \big(\boldsymbol{a}\,\sin(\beta t) + \boldsymbol{b}\,\cos(\beta t)\big)\,e^{\alpha t}.
\end{aligned}$$

$\square$

Example 4: Find real-valued fundamental solutions to the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Fist find the eigenvalues of matrix A above,

$$0 = \begin{vmatrix} (2-\lambda) & 3 \\ -3 & (2-\lambda) \end{vmatrix} = (\lambda - 2)^2 + 9 \quad \Rightarrow \quad \lambda_{\pm} = 2 \pm 3i.$$

Then find the respective eigenvectors. The one corresponding to λ_+ is the solution of the homogeneous linear system with coefficients given by

$$\begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix} = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}.$$

Therefore the eigenvector $\boldsymbol{v}^{(+)} = \begin{bmatrix} v_1^+ \\ v_2^+ \end{bmatrix}$ is given by

$$v_1^{(+)} = -iv_2^{(+)} \quad \Rightarrow \quad v_2^{(+)} = 1, \quad v_1^{(+)} = -i, \quad \Rightarrow \quad \boldsymbol{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \lambda_+ = 2 + 3i.$$

The second eigenvector is the complex conjugate of the eigenvector found above, that is,

$$\boldsymbol{v}^{(-)} = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \lambda_- = 2 - 3i.$$

Notice that

$$\boldsymbol{v}^{(\pm)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i.$$

Then, the real and imaginary parts of the eigenvalues and of the eigenvectors are given by

$$\alpha = 2, \qquad \beta = 3, \qquad \boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

So a real-valued expression for a fundamental set of solutions is given by

$$\boldsymbol{x}^{(1)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \right) e^{2t} \quad \Rightarrow \quad \boldsymbol{x}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t},$$

$$\boldsymbol{x}^{(2)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(3t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(3t) \right) e^{2t} \quad \Rightarrow \quad \boldsymbol{x}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t}.$$

$\triangleleft$

6.1.4. Diagonalizable Systems: Repeated Eigenvalues.

Remark: 2×2 linear differential systems with a diagonalizable coefficient matrix with a repeated eigenvalue are very simple to solve.

Theorem 3. (Diagonalizable with Repeated Eigenvalues) Every 2×2 diagonalizable matrix with a repeated eigenvalue λ_0 must have the form

$$A = \lambda_0 I.$$

Proof of Theorem 3: Since matrix A diagonalizable, there exists a matrix P invertible such that $A = PDP^{-1}$. Since A is 2×2 with a repeated eigenvalue λ_0, then

$$D = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix} = \lambda_0 \, I_2.$$

Put these two fatcs together,

$$A = P\lambda_0 I P^{-1} = \lambda_0 P \, P^{-1} = \lambda_0 I.$$

$\square$

Remark: : The differential equation $\boldsymbol{x}' = \lambda_0 I \, \boldsymbol{x}$ is already decoupled.

$$\left.\begin{aligned} x_1' &= \lambda_0 \, x_1 \\ x_2' &= \lambda_0 \, x_2 \end{aligned}\right\} \quad \Rightarrow \quad \text{too simple.}$$

6.1.5. Non-Diagonalizable Systems: Repeated Eigenvalues.

Example 5: Find fundamental solutions to the system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$$

Solution: We start computing the eigenvalues of A.

$$p(\lambda) = \begin{vmatrix} -6-\lambda & 4 \\ -1 & -2-\lambda \end{vmatrix} = (\lambda+6)(\lambda+2)+4 = \lambda^2+8\lambda+16 = (\lambda+4)^2.$$

We have a repeated eigenvalue $\lambda_0 = -4$. The eigenvector $\boldsymbol{v}$ is the solution of $(A+4I)\boldsymbol{v} = \boldsymbol{0}$,

$$\begin{bmatrix} -6+4 & 4 \\ -1 & -2+4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \Rightarrow \quad \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So we have only one equation

$$v_1 = 2v_2 \quad \Rightarrow \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_2$$

and choosing $v_2 = 1$ we get the eigenpair $\lambda_0 = -4$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So one fundamental solution is

$$\boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-4t}.$$

However, we do not know what is a second fundamental solution in this case.

$\triangleleft$

Theorem 4. (Non-Diagonalizable with a Repeated Eigenvalue) If a 2×2 matrix A has a repeated eigenvalue λ_0 with only one eigen direction determined by $\boldsymbol{v}_0$, then $\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t)$ has the linearly independent solutions

$$\boldsymbol{x}_1(t) = e^{\lambda t}\,\boldsymbol{v}, \qquad \boldsymbol{x}_2(t) = e^{\lambda t}\,\big(\boldsymbol{v}\,t + \boldsymbol{w}\big),$$

where the vector $\boldsymbol{w}$ is one solution of the algebraic linear system

$$(A - \lambda I)\boldsymbol{w} = \boldsymbol{v}.$$

Example 5-continued: Find the fundamental solutions of the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution: We already know that an eigenpair of A is

$$\lambda = -1, \qquad \boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Any other eigenvector associated to $\lambda = -1$ is proportional to the eigenvector above. The matrix A is not diagonalizable, so we solve for a vector $\boldsymbol{w}$ the linear system $(A + 4I)\boldsymbol{w} = \boldsymbol{v}$,

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad -w_1 + 2w_2 = 1 \quad \Rightarrow \quad w_1 = 2w_2 - 1.$$

Therefore,

$$\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2w_2 - 1 \\ w_2 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

So, given any solution $\boldsymbol{w}$, the $c\boldsymbol{v} + \boldsymbol{w}$ is also a solution for any $c \in \mathbb{R}$. We choose $w_2 = 0$,

$$\boldsymbol{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Therefore, a fundamental set of solutions to the differential equation above is formed by

$$\boldsymbol{x}_1(t) = e^{-4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{x}_2(t) = e^{-4t} \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right).$$

$\triangleleft$

6.2. Two-Dimensional Phase Portraits

Section Objective(s):

- Real Distinct Eigenvalues.
  - $\lambda_- < \lambda_+ < 0$, Sink (Stable).
  - $0 < \lambda_- < \lambda_+$, Source (Unstable).
  - $\lambda_- < 0 < \lambda_+ 0$, Saddle (Unstable).
- Complex Eigenvalues.

6.2.1. Review.

Theorem 1. The solutions of $\boldsymbol{x}' = A\,\boldsymbol{x}$, with A a 2×2 matrix, depend on the eigenpairs of A, say $\lambda_\pm$, $\boldsymbol{v}_\pm$, as follows.

(a) If $\lambda_+ \neq \lambda_-$ and real, then A is diagonalizable and

$$\boldsymbol{x}_+(t) = \boldsymbol{v}_+\, e^{\lambda_+ t}, \qquad \boldsymbol{x}_-(t) = \boldsymbol{v}_-\, e^{\lambda_- t},$$

(b) If $\lambda_+ = \alpha \pm \beta i$ and $\boldsymbol{v}_\pm = \boldsymbol{a} \pm \boldsymbol{b} i$, then A is diagonalizable and

$$\boldsymbol{x}_1(t) = \big(\boldsymbol{a}\cos(\beta t) - \boldsymbol{b}\sin(\beta t)\big)\, e^{\alpha t},$$

$$\boldsymbol{x}_2(t) = \big(\boldsymbol{a}\sin(\beta t) + \boldsymbol{b}\cos(\beta t)\big)\, e^{\alpha t}.$$

(c) If $\lambda_+ = \lambda_- = \lambda_0$ and A is diagonalizable, then $A = \lambda_0 I$ and

$$\boldsymbol{x}_+(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\lambda_0 t}, \qquad \boldsymbol{x}_-(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{\lambda_0 t},$$

(d) If $\lambda_+ = \lambda_- = \lambda_0$ and A is ***not*** diagonalizable, then

$$\boldsymbol{x}_+(t) = \boldsymbol{v}\, e^{\lambda_0 t}, \qquad \boldsymbol{x}_-(t) = (t\,\boldsymbol{v} + \boldsymbol{w})\, e^{\lambda_0 t},$$

where

$$(A - \lambda_0 I)\boldsymbol{v} = \boldsymbol{0}, \qquad (A - \lambda_0 I)\boldsymbol{w} = \boldsymbol{v}.$$

Example 1: Sketch a phase portrait and component plots of the of fundamental solutions of $\boldsymbol{x}' = A\,\boldsymbol{x}$, where the matrix A is given by

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

Solution: The eigenpairs of the matrix are $\lambda_{\pm} = \pm 2i$ and $\boldsymbol{v}_{\pm} = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}$, thus the real-valued fundamental solutions are

$$\boldsymbol{x}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(2t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(2t) = \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$$

$$\boldsymbol{x}_2(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(2t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(2t) = \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

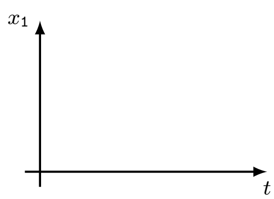
So, both solutions satisfy

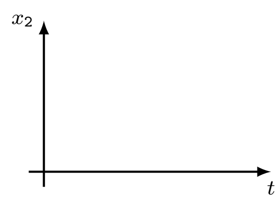
$$\|\boldsymbol{x}_1(t)\| = \sqrt{\cos^2(2t) + \sin^2(2t)} = 1, \qquad \|\boldsymbol{x}_2(t)\| = \sqrt{\sin^2(2t) + \cos^2(2t)} = 1,$$

so their curve in the x_1x_2-plane is a (part of a) circle radius 1, centered at the origin. The initial points are

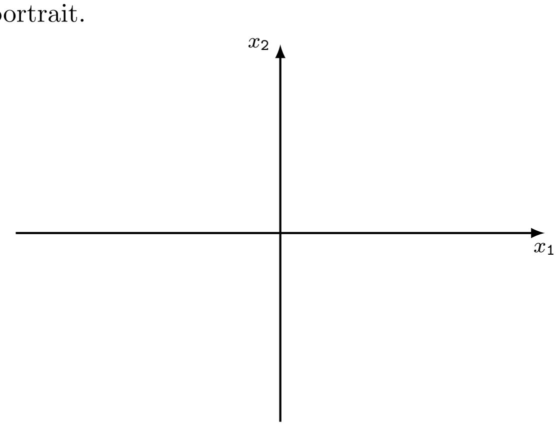
$$\boldsymbol{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- Graph each component of $\boldsymbol{x}_1(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ as function of t.





- Sketch a phase portrait.



Check the following **Interactive Graph**.

6.2.2. Real Distinct Eigenvalues.

Case $\lambda_- < \lambda_+ < 0$: ***Sink (Stable)***

Example 2: Sketch a phase portrait of the solutions of the system,

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix},$$

Hint: The eigenpairs of this matrix are $\lambda_1 = -4$, $\boldsymbol{v}_1 = \langle 1, 1\rangle$, and $\lambda_2 = -1$, $\boldsymbol{v}_2 = \langle -2, 1\rangle$.

Solution:

$$\boldsymbol{x}(t) = c_1\, e^{-4t}\, \boldsymbol{v}_1 + c_2\, e^{-t}\, \boldsymbol{v}_2 = c_1\, e^{-4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\, e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

- First plot the fundamental solutions: $\boldsymbol{x}_1(t) = e^{-4t}\, \boldsymbol{v}_1$ and $\boldsymbol{x}_2(t) = e^{-t}\, \boldsymbol{v}_2$.
- Then choose one more solution to plot: $c_1 = c_2 = 1$, that is.

$$\boldsymbol{x}(t) = e^{-4t}\, \boldsymbol{v}_1 + e^{-t}\, \boldsymbol{v}_2.$$

  - Find $\boldsymbol{x}(0) = \boldsymbol{v}_1 + \boldsymbol{v}_2$.
  - For $t \gg 1$ we get $x_1(t) \to 0$ and $x_2(t) \to 0$, but such that $\boldsymbol{x}(t) \to e^{-t}\, \boldsymbol{v}_2$.
  - For $t \ll -1$ we get $x_1(t) \to \infty$ and $x_2(t) \to \infty$, but such that $\boldsymbol{x}(t) \to e^{4|t|}\, \boldsymbol{v}_1$.

Check the following **Interactive Graph**.

Remark: Use the **Interactive Graph** to find the phase portraits of the solutions to the following cases:

Case $0 < \lambda_- < \lambda_+$ ***Source (Unstable)***

Example 3: Find the phase portrait of the solutions of the system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix},$$

Hint: The eigenpairs of this matrix are $\lambda_1 = 4$, $\boldsymbol{v}_1 = \langle 1, 1 \rangle$, and $\lambda_2 = 1$, $\boldsymbol{v}_2 = \langle -2, 1 \rangle$.

Case $\lambda_- < 0 < \lambda_+$ ***Saddle (Unstable)***

Example 4: Find the phase portrait of the solutions of the system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} -2 & -3 \\ -3 & -2 \end{bmatrix},$$

Hint: The eigenpairs of this matrix are $\lambda_1 = -5$, $\boldsymbol{v}_1 = \langle 1, 1 \rangle$, and $\lambda_2 = 1$, $\boldsymbol{v}_2 = \langle -1, 1 \rangle$.

6.2.3. Complex Eigenvalues.

Case $\lambda_{\pm} = \alpha \pm \beta i$: ***Spiral (Ellipse if*** $\alpha = 0$***).***

- $\alpha > 0$, **Source (Unstable).**
- $\alpha = 0$, **Center.**
- $\alpha < 0$, **Sink (Stable).**

Remark: Use the **Interactive Graph** to help understand the phase portraits of the solutions to the following example.

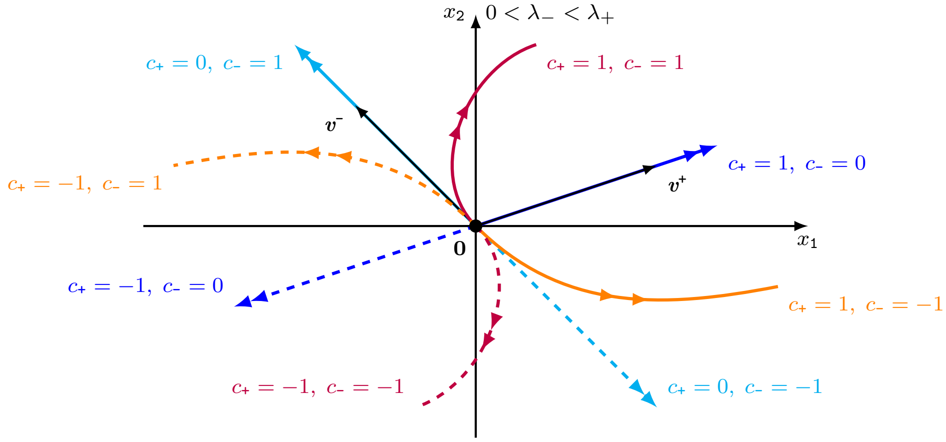
Example 5: Find the phase portrait of the solutions of the system

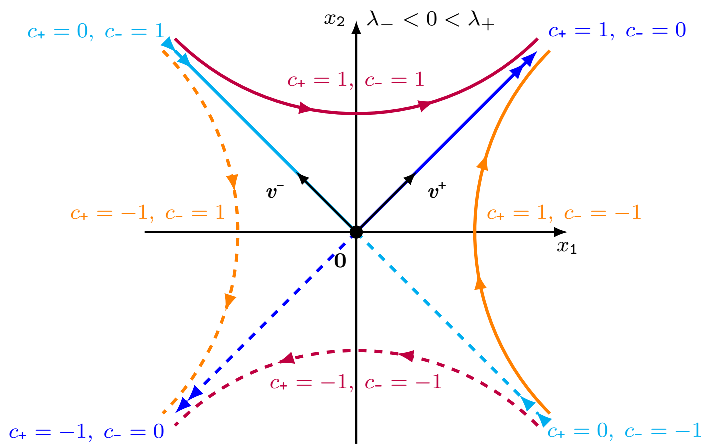
$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}.$$

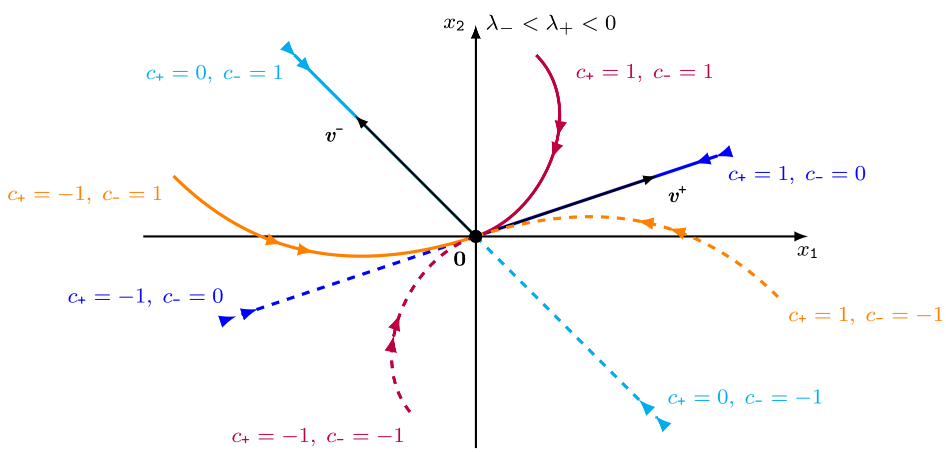
Hint: The eigenpairs of this matrix are $\lambda_{\pm} = -2 \pm 3i$, $\boldsymbol{v}_{\pm} = \langle \pm i, 1 \rangle$.

$$\boldsymbol{x}(t) = c_1\, e^{-2t} \begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix} + c_2\, e^{-2t} \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix}$$

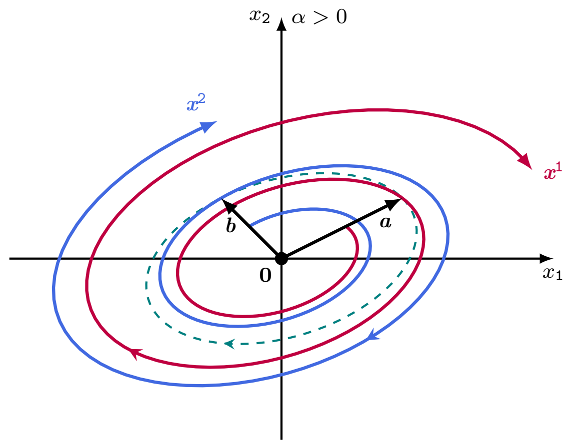
Remark: Summary:

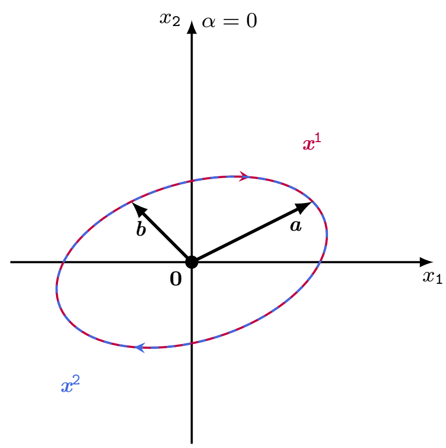


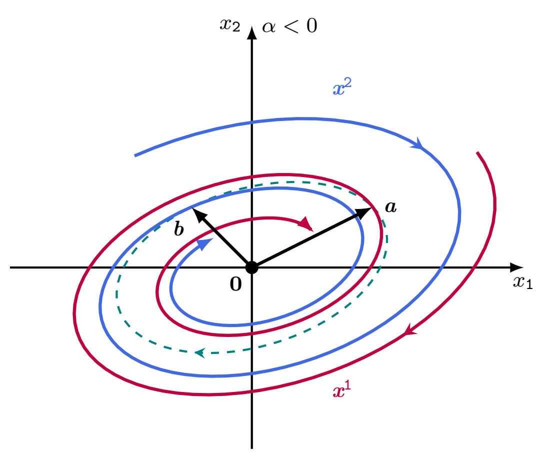




Remark: Summary:







6.3. Nonlinear Systems of Equations

Section Objective(s):

Part One:

- Two-Dimensional Nonlinear Systems.
- Critical Points and Linearization.
- The Hartman-Grobman Theorem.

Part Two:

- Competing Species: Extinction.
- Competing Species: Coexistence.

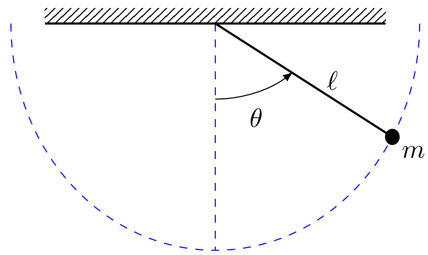
Remarks:

- We know how to solve systems of linear differential equations.
- But systems of nonlinear differential equations are harder to solve.
- In this section we find qualitative properties of the solutions to nonlinear systems.
- We first find the critical points of the nonlinear system.
- We then find the behavior of solutions to nonlinear systems near the critical points. (Linearizations.)
- Finally, we glue together the information from all the critical points to get a qualitative phase portrait of solutions to the nonlinear system.
- We focus on two versions of the competing species system:
  - The case when one species goes extinct.
  - The case when both species coexist.

6.3.1. Two-Dimensional Nonlinear Systems.

Example 1: (The Nonlinear Pendulum)

$$m(\ell\theta)'' = -mg\sin(\theta),$$



that is

$$\theta'' + \frac{g}{\ell}\sin(\theta) = 0.$$

Introduce $x_1 = \theta$ and $x_2 = \theta'$,

$$x_1' = x_2$$

$$x_2' = -\frac{g}{\ell}\sin(x_1).$$

$\triangleleft$

Example 2: (Predator-Prey)
Let x be the predator and y be the prey. Then, the equation is

$$x_1' = -a\,x_1 + b\,x_1 x_2,$$

$$x_2' = -c\,x_1 x_2 + d\,x_2.$$

$\triangleleft$

Example 3: (Competing Species)
Let x_1 be the rabbit population and x_2 be the sheep population, both competing for the same food resources. The equation is

$$x_1' = r_1\,x_1\left(1 - \frac{x_1}{K_1} - \alpha\,x_2\right),$$

$$x_2' = r_2\,x_2\left(1 - \frac{x_2}{K_2} - \beta\,x_1\right).$$

$\triangleleft$

6.3.2. Critical Points and Linearization.

Definition 1. A critical point of a system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ is the end point of a vector $\boldsymbol{x}_c$ solution of

$$\boldsymbol{f}(\boldsymbol{x}_c) = \boldsymbol{0}.$$

Remarks:

(a) Recall that $x = (x_1, x_2)$ is a point on the x_1x_2-plane while $\boldsymbol{x} = \langle x_1, x_2 \rangle$ is a vector with origin at $(0,0)$ and end point at $x = (x_1, x_2)$.

(b) $\boldsymbol{x}_c$ is solution of $\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x})$, since

$$(\boldsymbol{x}_c)' = \boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x}_c).$$

(c) In components, the field is $\boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, and the vector $\boldsymbol{x}_c = \begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix}$ is solution of

$$f_1(x_{c1}, x_{c2}) = 0,$$

$$f_2(x_{c1}, x_{c2}) = 0.$$

When there are more than one critical point we write $\boldsymbol{x}_{c_i}$, with $i = 0, 1, 2, \cdots$.

Example 4: Find all the critical points of the two-dimensional (decoupled) system

$$x_1' = -x_1 + (x_1)^3$$
$$x_2' = -2\,x_2.$$

Solution: We need to find all constant vectors $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ solutions of

$$-x_1 + (x_1)^3 = 0, \qquad -2\,x_2 = 0.$$

From the second equation we get $x_2 = 0$. From the first equation we get

$$x_1\big((x_1)^2 - 1\big) = 0 \quad \Rightarrow \quad x_1 = 0, \quad \text{or} \quad x_1 = \pm 1.$$

Therefore, we got three critical points, $\boldsymbol{x}^{c_0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\boldsymbol{x}^{c_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{x}^{c_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

◁

Definition 2. The linearization ____________ of a 2×2 system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ at a critical point given by $\boldsymbol{x}_c$ is the 2×2 linear system

$$\boldsymbol{u}' = (Df_c)\,\boldsymbol{u},$$

where the Jacobian matrix ____________ at $\boldsymbol{x}_c$ is,

$$Df_c = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{\boldsymbol{x}_c} & \left.\frac{\partial f_1}{\partial x_2}\right|_{\boldsymbol{x}_c} \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{\boldsymbol{x}_c} & \left.\frac{\partial f_2}{\partial x_2}\right|_{\boldsymbol{x}_c} \end{bmatrix} = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{bmatrix}.$$

Remark: : In components, the nonlinear system its linearization are

$$\begin{aligned} x_1' &= f_1(x_1, x_2), \\ x_2' &= f_2(x_1, x_2), \end{aligned}, \qquad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Example 5: Find the linearization at every critical point of the nonlinear system

$$\begin{aligned} x_1' &= -x_1 + (x_1)^3 \\ x_2' &= -2\,x_2. \end{aligned}$$

Solution: We found earlier that this system has three critial points,

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

This means we need to compute three linearizations, one for each critical point. We start computing the derivative matrix at an arbitrary point $\boldsymbol{x}$,

$$Df(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1}(-x_1 + x_1^3) & \frac{\partial}{\partial x_2}(-x_1 + x_1^3) \\ \frac{\partial}{\partial x_1}(-2x_2) & \frac{\partial}{\partial x_2}(-2x_2) \end{bmatrix},$$

so we get that

$$Df(\boldsymbol{x}) = \begin{bmatrix} -1 + 3\,x_1^2 & 0 \\ 0 & -2 \end{bmatrix}.$$

We only need to evaluate this matrix Df at the critical points. We start with $\boldsymbol{x}_0$,

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad Df_0 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The Jacobian at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is the same, so we get the same linearization at these points,

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad Df_1 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad Df_2 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$\triangleleft$

6.3.3. The Hartman-Grobman Theorem.

Remark: The linearization of a nonlinear system allow us to classify the critical points of nonlinear systems. linearization.

Definition 3. A critical point $\boldsymbol{x}^c$ of a 2×2 system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ is:

(a) an ***sink*** iff both eigenvalues of Df_c have negative real part;

(b) a ***source*** iff both eigenvalues of Df_c have positive real part;

(c) a ***saddle*** iff one eigenvalue of Df_c is positive and the other is negative;

(d) a ***center*** iff both eigenvalues of Df_c are pure imaginary;

A critical point $\boldsymbol{x}^c$ is called ***hyperbolic*** iff it belongs to cases (a-c), that is, the real part of all eigenvalues of Df_c are nonzero.

Theorem 1. (Hartman-Grobman) Consider a 2×2 nonlinear autonomous system,

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}),$$

with $\boldsymbol{f}$ continuously differentiable, and consider its linearization at a hyperbolic critical point given by $\boldsymbol{x}_c$,

$$\boldsymbol{u}' = (Df_c)\, \boldsymbol{u}.$$

Then, there is a neighborhood of $\boldsymbol{x}_c$ where all the solutions of the linear system can be transformed into solutions of the nonlinear system by a continuous, invertible, transformation.

Remark: The theorem above says that the phase portrait of the linearization at a hyperbolic critical point is enough to determine the qualitative picture of the phase portrait of the nonlinear system near that critical point.

Example 6: Use the Hartman-Grobman theorem to sketch the phase portrait of

$$x_1' = -x_1 + (x_1)^3$$
$$x_2' = -2\,x_2.$$

Solution: We already know that this system has three critical points,

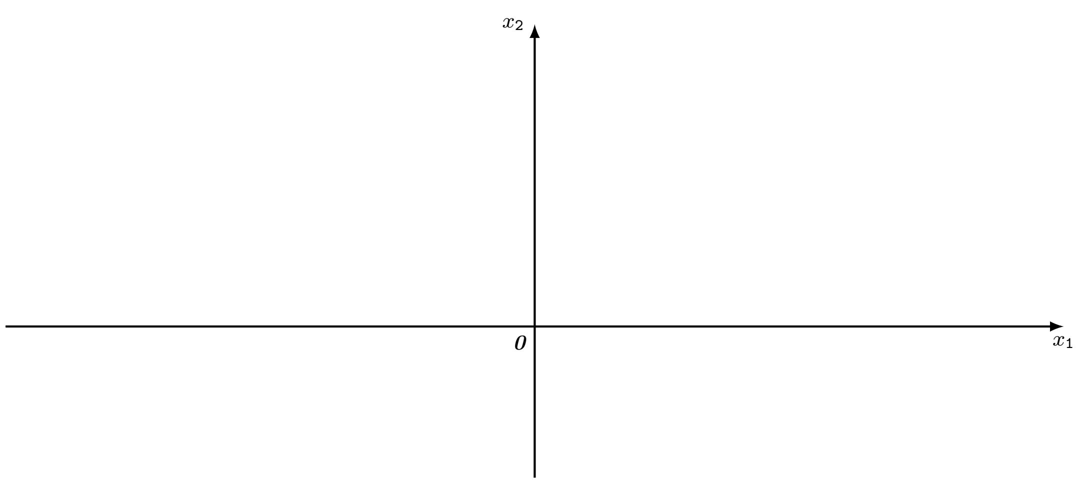
$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

We have already computed the linearizations at these critical points too.

$$Df_0 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad Df_1 = Df_2 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

We now need to compute the eigenvalues of the Jacobian matrices above.

- For the critical point $\boldsymbol{x}_0$ we have $\lambda_+ = -1$, $\lambda_- = -2$, so $\boldsymbol{x}_0$ is an attractor.
- For the critical points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ we have $\lambda_+ = 2$, $\lambda_- = -2$, so $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are saddle points.



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6.3.4. Competing Species: Extinction.

Example 7: Find the linearization at every critical point of the competing species system

$$r' = r\,(3 - r - 2\,s),$$
$$s' = s\,(2 - s - r),$$

Remark: We call this model a rabbits-sheep model, where $r(t)$ is the rabbit population and $s(t)$ is the sheep population at the time t.

Solution: We start finding all the critical points of the rabbit-sheep system.

$$r\,(3 - r - 2\,s) = 0,$$
$$s\,(2 - s - r) = 0.$$

There are four solutions to the equations above:

(1) $r = 0$ and $s = 0$;
(2) $r = 0$ and $2 - s - r = 0$;
(3) $3 - r - 2\,s = 0$ and $s = 0$;
(4) $3 - r - 2\,s = 0$ and $2 - s - r = 0$.

From these equations we get

(1) $(r = 0, s = 0)$;
(2) $(r = 0, s = 2)$;
(3) $(r = 3, s = 0)$;
(4) the intersection of the lines $s = (3 - r)/2$ and $s = (2 - r)$ which is given by

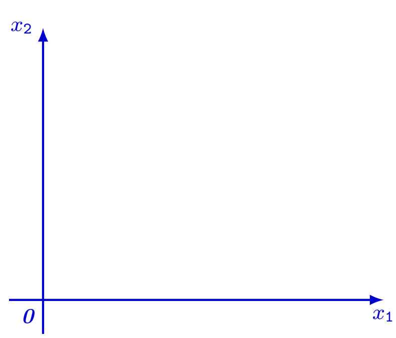
$$\frac{3 - r}{2} = 2 - r \quad \Rightarrow \quad 3 - r = 4 - 2r \quad \Rightarrow \quad r = 1, \quad \Rightarrow \quad (r = 1, s = 1).$$

Summarizing, we got the four critical points

$$x_0 = (0, 0), \qquad x_1 = (0, 2), \qquad x_2 = (3, 0), \qquad x_3 = (1, 1).$$

we can always think the points as the end points of the vectors

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$



Now we find the linearization of the rabbit-sheep system. If $\boldsymbol{x} = \begin{bmatrix} r \\ s \end{bmatrix}$, the system is $\boldsymbol{x}' = \boldsymbol{F}(\boldsymbol{x})$,

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} r\,(3 - r - 2\,s) \\ s\,(2 - s - r) \end{bmatrix}.$$

The derivative of $\boldsymbol{F}$ at an arbitrary point $\boldsymbol{x}$ is

$$DF(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial r} & \frac{\partial F_1}{\partial s} \\ \frac{\partial F_2}{\partial r} & \frac{\partial F_2}{\partial s} \end{bmatrix} = \begin{bmatrix} (3 - 2\,r - 2\,s) & -2\,r \\ -s & (2 - 2\,s - r) \end{bmatrix}.$$

We now evaluate the matrix $DF(\boldsymbol{x})$ at each of the critical points we found.

(0)

$$\text{At} \quad \boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{we get} \quad (DF_0) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \lambda_{0+} = 3 \\ \lambda_{0-} = 2. \end{cases}$$

The critical point x_0 is a ***source node***. To sketch the phase portrait we will need the corresponding eigenvectors, $\boldsymbol{v}_0^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_0^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(1)

$$\text{At} \quad \boldsymbol{x}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \text{we get} \quad (Df_1) = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \lambda_{0+} = -1 \\ \lambda_{0-} = -2. \end{cases}$$

The critical point x_1 is an ***sink node***. One can check that the corresponding eigenvectors are $\boldsymbol{v}_1^+ = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\boldsymbol{v}_1^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(2)

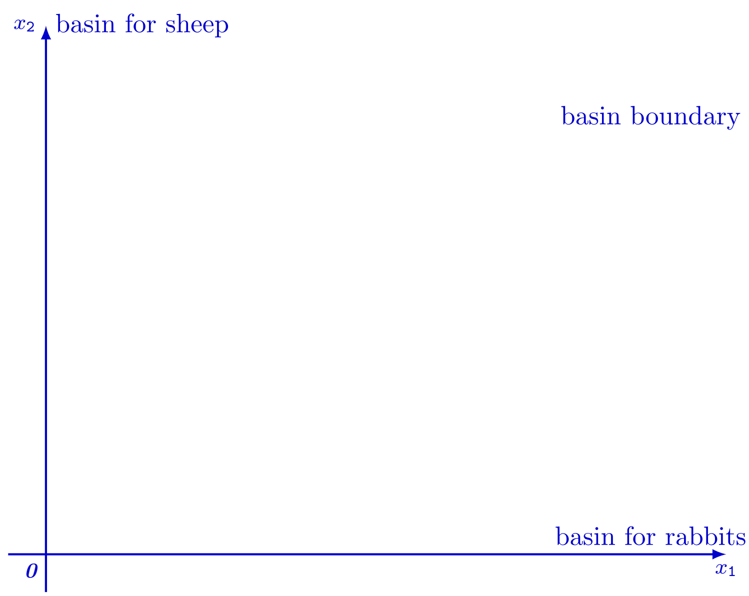
$$\text{At} \quad \boldsymbol{x}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{we get} \quad (Df_2) = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \lambda_{0+} = -1 \\ \lambda_{0-} = -3. \end{cases}$$

The critical point x_2 is an ***source node***. One can check that the corresponding eigenvectors are $\boldsymbol{v}_2^+ = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_2^- = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(3)

$$\text{At} \quad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{we get} \quad (Df_3) = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \lambda_{0+} = -1 + \sqrt{2} \\ \lambda_{0-} = -2 - \sqrt{2}. \end{cases}$$

The critical point x_3 is a ***saddle node***. One can check that the corresponding eigenvectors are $\boldsymbol{v}_3^+ = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_3^- = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$.



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6.3.5. Competing Species: Coexistence.

Example 7: Find the linearization at every critical point of the competing species system

$$r' = r\,(1 - r - s),$$
$$s' = \frac{s}{4}\,(3 - 4\,s - 2\,r),$$

Remark: This is also a rabbits-sheep model, where $r(t)$ is the rabbit population and $s(t)$ is the sheep population at the time t.

Solution: Th equation for the critical points are

$$r\,(1 - r - s) = 0,$$
$$\frac{s}{4}\,(3 - 4\,s - 2\,r) = 0.$$

Check that the critical points for this system are

$$x_0 = (0, 0), \qquad x_1 = \Big(0, \frac{3}{4}\Big), \qquad x_2 = (1, 0), \qquad x_3 = \Big(\frac{1}{2}, \frac{1}{2}\Big).$$

The fector field of this system is

$$\boldsymbol{F} = \begin{bmatrix} r\,(1 - r - s) \\ \frac{1}{4}\,s\,(3 - 4\,s - 2\,r) \end{bmatrix}$$

The derivative of $\boldsymbol{F}$ is

$$DF(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial r} & \frac{\partial F_1}{\partial s} \\ \frac{\partial F_2}{\partial r} & \frac{\partial F_2}{\partial s} \end{bmatrix} = \begin{bmatrix} (1 - 2\,r - s) & -r \\ -\frac{1}{2}\,s & (\frac{3}{4} - 2\,s - \frac{1}{2}\,r) \end{bmatrix}.$$

Then, one can check that the critical points above satisfy the following:

(0)

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (DF_0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \lambda_{0+} = 1, \quad \boldsymbol{v}_0^{+} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \lambda_{0-} = \dfrac{3}{4}, \quad \boldsymbol{v}_0^{+} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{cases}$$

(1)

$$\boldsymbol{x}_1 = \begin{bmatrix} 0 \\ \frac{3}{4} \end{bmatrix}, \quad (DF_1) = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \lambda_{1+} = \dfrac{1}{4}, & \boldsymbol{v}_1^{+} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}, \\ \lambda_{1-} = -\dfrac{3}{4}, & \boldsymbol{v}_1^{+} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{cases}$$

(2)

$$\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (DF_2) = \begin{bmatrix} -1 & -1 \\ 0 & \frac{1}{4} \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \lambda_{2+} = \dfrac{1}{4}, & \boldsymbol{v}_2^{+} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}, \\ \lambda_{2-} = -1, & \boldsymbol{v}_2^{+} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

$$\boldsymbol{x}_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad (DF_3) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \lambda_{3+} = \dfrac{1}{4}(-2+\sqrt{2}), & \boldsymbol{v}_3^{+} = \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}, \\ \lambda_{3-} = \dfrac{1}{4}(-2-\sqrt{2}), & \boldsymbol{v}_3^{+} = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}. \end{cases}$$

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7.1. Simple Eigenfunction Problems

Section Objective(s):
- Two-Point Boundary Value Problems.
- Comparing IVP vs BVP.
- Eigenfunction Problems.

Remark:

- The main idea of this chapter is to solve the heat equation.
- This is a partial differential equation.
- We need two main ideas to solve that equation.
  - **(1)** Boundary value problems and eigenfunction problems.
  - **(2)** Fourier series expansions.
- In this section we study the first idea: boundary value problems and eigenfunctions.

7.1.1. Two-Point Boundary Value Problems.

Definition 1. A ***two-point boundary value problem*** (BVP) is the following: Find solutions to the differential equation

$$y'' + a_1(x)\, y' + a_0(x)\, y = b(x)$$

satisfying the boundary conditions (BC)

$$b_1\, y(x_1) + b_2\, y'(x_1) = y_1,$$

$$\tilde{b}_1\, y(x_2) + \tilde{b}_2\, y'(x_2) = y_2,$$

where b_1, b_2, $\tilde{b}_1$, $\tilde{b}_2$, y_1, y_2, x_1, x_2 are given and $x_1 \neq x_2$.

Remarks:

(a) The two boundary conditions are held at different points, $\underline{x_1 \neq x_2}$.

(b) Both $\underline{y \text{ and } y'}$ may appear in the boundary condition.

Example 1: We now show four examples of boundary value problems that differ only on the boundary conditions: Solve the different equation

$$y'' + a_1\, y' + a_0\, y = b(x)$$

with the boundary conditions at $x_1 = 0$ and $x_2 = 1$ given below.

(1a)

Boundary Condition: $\left\{\begin{matrix} y(0) = y_1, \\ y(1) = y_2, \end{matrix}\right\}$ which is the case $\left\{\begin{matrix} b_1 = \underline{1}, & b_2 = \underline{0}, \\ \tilde{b}_1 = \underline{1}, & \tilde{b}_2 = \underline{0}. \end{matrix}\right\}$

(1b)

Boundary Condition: $\left\{\begin{matrix} y(0) = y_1, \\ y'(1) = y_2, \end{matrix}\right\}$ which is the case $\left\{\begin{matrix} b_1 = \underline{1}, & b_2 = \underline{0}, \\ \tilde{b}_1 = \underline{0}, & \tilde{b}_2 = \underline{1}. \end{matrix}\right\}$

(1c)

Boundary Condition: $\left\{\begin{matrix} y'(0) = y_1, \\ y(1) = y_2, \end{matrix}\right\}$ which is the case $\left\{\begin{matrix} b_1 = \underline{0}, & b_2 = \underline{1}, \\ \tilde{b}_1 = \underline{1}, & \tilde{b}_2 = \underline{0}. \end{matrix}\right\}$

(1d)

Boundary Condition: $\left\{\begin{matrix} y'(0) = y_1, \\ y'(1) = y_2, \end{matrix}\right\}$ which is the case $\left\{\begin{matrix} b_1 = \underline{0}, & b_2 = \underline{1}, \\ \tilde{b}_1 = \underline{0}, & \tilde{b}_2 = \underline{1}. \end{matrix}\right\}$

$\triangleleft$

7.1.2. Comparing IVP vs BVP.

Definition 2. (IVP) Find a solution of $y'' + a_1\, y' + a_0\, y = 0$ satisfying the initial condition (IC)

$$y(t_0) = y_0, \qquad y'(t_0) = y_1.$$

Remarks:

- The variable t represents time.
- The variable y represents position.
- The IC are position and velocity at the initial time.

Definition 3. (BVP) Find a solution y of $y'' + a_1\, y' + a_0\, y = 0$ satisfying the boundary condition (BC)

$$y(x_0) = y_0, \qquad y(x_1) = y_1.$$

Remarks:

- The variable x represents position.
- The variable y may represent temperature.
- The BC are temperature at two different positions.

Theorem 1. The equation $y'' + a_1\, y' + a_0\, y = 0$ with IC $y(t_0) = y_0$ and $y'(t_0) = y_1$ has a unique solution y for each choice of the IC.

Theorem 2. (BVP) The equation $y'' + a_1\, y' + a_0\, y = 0$ with BC $y(0) = y_0$ and $y(L) = y_1$, with $L \neq 0$ and with $r_\pm$ roots of $p(r) = r^2 + a_1 r + a_0$ satisfy the following:

(A) If $r_+ \neq r_-$, reals, then the BVP above has a unique solution.

(B) If $r_\pm$ are complex, then the solution of the BVP above belongs to only one of the following three possibilities:

(i) There exists a unique solution.

(ii) There exists infinitely many solutions.

(iii) There exists no solution.

Proof of Theorem 2: The general solution is

$$y(x) = c_+ \, e^{t_+ x} + c_- \, e^{r_- x}.$$

The BC are

$$\left.\begin{aligned} y_0 &= y(0) = c_+ + c_- \\ y_1 &= y(L) = c_+ \, e^{c_+ L} + c_- \, e^{c_- L} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

This system for c_+, c_- has a unique solution iff

$$0 \neq \begin{vmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{vmatrix} \quad \Rightarrow \quad e^{r_- L} - e^{r_+ L} \neq 0.$$

Part (A): If $r_+ \neq r_-$, reals, then $e^{r_- L} \neq e^{r_+ L}$, hence there is a unique solution c_+, c_-, which fixes a unique solution y of the BVP.

Part (B): If $r_\pm = \alpha \pm i\beta$, then

$$e^{r_\pm L} = e^{(\alpha \pm i\beta)L} = e^{\alpha L}\big(\cos(\beta L) \pm i \sin(\beta L)\big),$$

therefore

$$\begin{aligned} e^{r_- L} - e^{r_+ L} &= e^{\alpha L}\big(\cos(\beta L) - i\sin(\beta L) - \cos(\beta L) - i \sin(\beta L)\big) \\ &= -2i \, e^{\alpha L} \, \sin(\beta L) = 0 \quad \Leftrightarrow \quad \beta L = n\pi. \end{aligned}$$

So for $\beta L \neq n\pi$ the BVP has a unique solution, case (Bi).

For $\beta L = n\pi$ the BVP has either no solution or infinitely many solutions, cases (Bii) and (Biii).

$\square$

Example 2: Find all solutions to the BVPs $y'' + y = 0$ with the BCs:

$$\text{(a)} \quad \begin{cases} y(0) = 1, \\ y(\pi) = 0. \end{cases} \qquad \text{(b)} \quad \begin{cases} y(0) = 1, \\ y(\pi/2) = 1. \end{cases} \qquad \text{(c)} \quad \begin{cases} y(0) = 1, \\ y(\pi) = -1. \end{cases}$$

Solution: We first find the roots of the characteristic polynomial $r^2 + 1 = 0$, that is, $r_{\pm} = \pm i$. So the general solution of the differential equation is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

BC (a):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$0 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 0.$$

Therefore, there is no solution.

BC (b):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$1 = y(\pi/2) = c_2 \quad \Rightarrow \quad c_2 = 1.$$

So there is a unique solution $y(x) = \cos(x) + \sin(x)$.

BC (c):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$-1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_2 = 1.$$

Therefore, c_2 is arbitrary, so we have infinitely many solutions

$$y(x) = \cos(x) + c_2 \sin(x), \qquad c_2 \in \mathbb{R}.$$

$\triangleleft$

7.1.3. Eigenfunction Problems.

Remark: Let us recall the ***eigenvector*** problem of a square matrix: Given a square matrix A, find a number λ and a nonzero vector $\boldsymbol{v}$ solution of

$$A\boldsymbol{v} = \lambda\,\boldsymbol{v}.$$

Definition 4. An ***eigenfunction problem*** is the following: Given a linear operator $L(y) = a_2\,y'' + a_1\,y' + a_0\,y$, find a number λ and a nonzero function y solution of

$$L(y) = \lambda y,$$

and homogeneous boundary conditions at $x_1 \neq x_2$,

$$b_1\,y(x_1) + b_2\,y'(x_1) = 0,$$

$$\tilde{b}_1\,y(x_2) + \tilde{b}_2\,y'(x_2) = 0,$$

Remarks:

- Notice that $y = 0$ is always a solution of the BVP above.
- Eigenfunctions are the nonzero solutions of the BVP above.
- The eigenfunction problem is a BVP with infinitely many solutions.
- So, we look for λ such that the operator $L(y) - \lambda\,y$ has characteristic polynomial with complex roots.
- So, λ is such that $L(y) - \lambda\,y$ has oscillatory solutions.
- We focus on the linear operator $L(y) = -y''$.

Example 3: Find all numbers λ and nonzero functions y solutions of the BVP

$$-y'' = \lambda\,y, \qquad \text{with} \qquad y(0) = 0, \quad y(L) = 0, \qquad L > 0.$$

Solution:

The equation is $y' + \lambda\,y = 0$. We have three cases: **(a)** $\lambda < 0$, **(b)** $\lambda = 0$, and **(c)** $\lambda > 0$.

Case (a): $\lambda = -\mu^2 < 0$, so the equation is $y'' - \mu^2 y = 0$. The characteristic equation is

$$r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu.$$

The general solution is $y = c_+ e^{\mu x} + c_- e^{-\mu x}$. The BC imply

$$0 = y(0) = c_+ + c_-, \qquad 0 = y(L) = c_+ e^{\mu L} + c_- e^{-\mu L}.$$

So from the first equation we get $c_+ = -c_-$, so

$$0 = -c_- e^{\mu L} + c_- e^{-\mu L} \quad \Rightarrow \quad -c_-(e^{\mu L} - e^{-\mu L}) = 0 \quad \Rightarrow \quad c_- = 0, \quad c_+ = 0.$$

So we get only the solution $y = 0$.

Case (b): $\lambda = 0$, so the differential equation is

$$y'' = 0 \quad \Rightarrow \quad y = c_0 + c_1 x.$$

The BC imply

$$0 = y(0) = c_0, \qquad 0 = y(L) = c_1 L \quad \Rightarrow \quad c_1 = 0.$$

So we get the only solution is $y = 0$.

Case (c): $\lambda = \mu^2 > 0$, so the equation is $y'' + \mu^2 y = 0$. The characteristic equation is

$$r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu i.$$

The general solution is $y = c_+ \cos(\mu x) + c_- \sin(\mu x)$. The BC imply

$$0 = y(0) = c_+, \qquad 0 = y(L) = c_+ \cos(\mu L) + c_- \sin(\mu L) = c_- \sin(\mu L),$$

therefore, we get

$$c_- \sin(\mu L) = 0, \qquad c_- \neq 0 \quad \Rightarrow \quad \sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi.$$

So we get $\mu_n = n\pi/L$, hence the eigenvalue eigenfunction pairs are

$$\lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \qquad y_n(x) = c_n \sin\Big(\frac{n\pi x}{L}\Big).$$

$\lhd$

Example 4: Find the numbers λ and the nonzero functions y solutions of the BVP

$$-y'' = \lambda y, \qquad y(0) = 0, \qquad y'(L) = 0, \qquad L > 0.$$

Solution:

The equation is $y'' + \lambda y = 0$. We have three cases: **(a)** $\lambda < 0$, **(b)** $\lambda = 0$, and **(c)** $\lambda > 0$.

Case (a): Let $\lambda = -\mu^2$, with $\mu > 0$, so the equation is $y'' - \mu^2\, y = 0$. The characteristic equation is

$$r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu,$$

The general solution is $y(x) = c_1 e^{-\mu x} + c_2 e^{\mu x}$. The BC imply

$$\left.\begin{aligned} 0 &= y(0) = c_1 + c_2, \\ 0 &= y'(L) = -\mu c_1 e^{-\mu L} + \mu c_2 e^{\mu L} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ -\mu e^{-\mu L} & \mu e^{\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix above is invertible, because

$$\begin{vmatrix} 1 & 1 \\ -\mu e^{-\mu L} & \mu e^{\mu L} \end{vmatrix} = \mu\big(e^{\mu L} + e^{-\mu L}\big) \neq 0.$$

Therefore, the linear system above for c_1, c_2 has a unique solution given by $c_1 = c_2 = 0$. Hence, we get the only solution $y = 0$. This means there are no eigenfunctions with negative eigenvalues.

Case (b): Let $\lambda = 0$, so the differential equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$0 = y(0) = c_1, \qquad 0 = y'(L) = c_2.$$

So the only solution is $y = 0$. This means there are no eigenfunctions with eigenvalue $\lambda = 0$.

Case (c): Let $\lambda = \mu^2$, with $\mu > 0$, so the equation is $y'' + \mu^2\, y = 0$. The characteristic equation is

$$r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu\, i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$. The BC imply

$$\left.\begin{aligned} 0 &= y(0) = c_1, \\ 0 &= y'(L) = -\mu c_1 \sin(\mu L) + \mu c_2 \cos(\mu L) \end{aligned}\right\} \quad \Rightarrow \quad c_2 \cos(\mu L) = 0.$$

Since we are interested in non-zero solutions y, we look for solutions with $c_2 \neq 0$. This implies that μ cannot be arbitrary but must satisfy the equation

$$\cos(\mu L) = 0 \quad \Leftrightarrow \quad \mu_n L = (2n-1)\frac{\pi}{2}, \qquad n \geqslant 1.$$

We therefore conclude that the eigenvalues and eigenfunctions are given by

$$\lambda_n = -\frac{(2n-1)^2\pi^2}{4L^2}, \qquad y_n(x) = c_n \sin\Big(\frac{(2n-1)\pi x}{2L}\Big), \qquad n \geqslant 1.$$

Since we only need one eigenfunction for each eigenvalue, we choose $c_n = 1$, and we get

$$\lambda_n = -\frac{(2n-1)^2\pi^2}{4L^2}, \qquad y_n(x) = \sin\Big(\frac{(2n-1)\pi x}{2L}\Big), \qquad n \geqslant 1.$$

$\triangleleft$

7.2. Overview of Fourier series

Section Objective(s):

- Vectors and the Dot Product in $\mathbb{R}^n$.
- Fourier Expansion of Functions.
- Odd or Even Functions.
- Sine and Cosine Series.

Remarks:

- We start with the Fourier expansion of a vector in $\mathbb{R}^3$.
- We review a few concepts:
  - The dot product of two vectors.
  - Orthogonal and orthonormal vectors.
  - The decomposition of a vector in an orthonormal basis.
- We then introduce the Fourier expansion of a continuous function.
- We need the following concepts:
  - The dot product of two functions.
  - Orthogonal and orthonormal functions.
  - The decomposition of a function in an orthonormal basis.
- We finish with two particular cases, the Fourier expansion of even functions and of odd functions.

7.2.1. Vectors and the Dot Product in $\mathbb{R}^n$.

Remark: We review basic concepts about vectors in $\mathbb{R}^3$.

Definition 1. The ***dot product*** of $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$, $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Remark: The dot product above satisfies the following properties.

Theorem 1. For every $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbb{R}^3$ and every a, $b \in \mathbb{R}$ the following holds true:

(a) Positivity: $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ iff $\boldsymbol{u} = 0$; and $\boldsymbol{u} \cdot \boldsymbol{u} > 0$ for $\boldsymbol{u} \neq 0$.

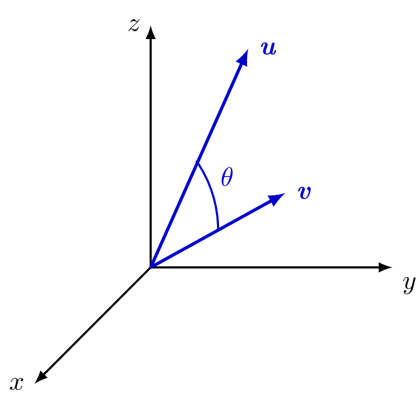
(b) Symmetry: $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$.

(c) Linearity: $(a\boldsymbol{u} + b\boldsymbol{v}) \cdot \boldsymbol{w} = a\,(\boldsymbol{u} \cdot \boldsymbol{w}) + b\,(\boldsymbol{v} \cdot \boldsymbol{w})$.

Theorem 2. The ***dot product*** of two vectors $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{R}^3$ is

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos(\theta),$$

with $\|\boldsymbol{u}\|$, $\|\boldsymbol{v}\|$ the magnitude of the vectors, and $\theta \in [0, \pi]$ the angle in between them.

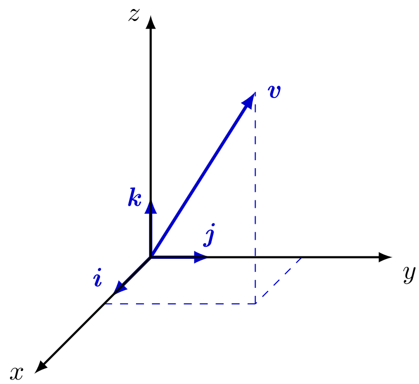


Remarks:

- The magnitude of a vector $\boldsymbol{u}$ can be written as
$$\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}.$$
- A vector $\boldsymbol{u}$ is a unit vector iff
$$\|\boldsymbol{u}\| = 1.$$

Theorem 3. The vectors $\boldsymbol{u}$, $\boldsymbol{v}$ are orthogonal iff $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

Example 1: The set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is an orthonormal set of $\mathbb{R}^3$.



Orthonormal means:

- Orthogonality:

$$\boldsymbol{i} \cdot \boldsymbol{j} = 0, \quad \boldsymbol{i} \cdot \boldsymbol{k} = 0, \quad \boldsymbol{j} \cdot \boldsymbol{k} = 0.$$

- Normality:

$$\boldsymbol{i} \cdot \boldsymbol{i} = 1, \quad \boldsymbol{j} \cdot \boldsymbol{j} = 1, \quad \boldsymbol{k} \cdot \boldsymbol{k} = 1.$$

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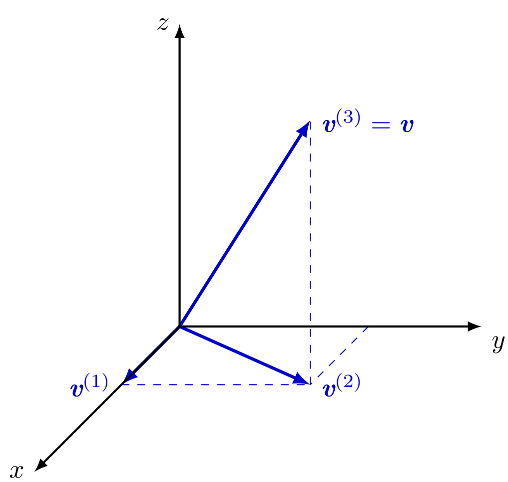
Theorem 4. (Fourier Expansion) The orthonormal set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is an orthonormal basis, that is, every $\boldsymbol{v} \in \mathbb{R}^3$ can be decomposed as

$$\boldsymbol{v} = v_x\,\boldsymbol{i} + v_y\,\boldsymbol{j} + v_z\,\boldsymbol{k}.$$

The orthonormality of the vector set implies a formula for the vector components

$$v_x = \boldsymbol{v} \cdot \boldsymbol{i}, \qquad v_y = \boldsymbol{v} \cdot \boldsymbol{j}, \qquad v_z = \boldsymbol{v} \cdot \boldsymbol{k}.$$

Remark: The decomposition above allows us to introduce vector approximations.



Vector Approximations:

$$\boldsymbol{v}^{(1)} = v_x\,\boldsymbol{i},$$

$$\boldsymbol{v}^{(2)} = v_x\,\boldsymbol{i} + v_y\,\boldsymbol{j},$$

$$\boldsymbol{v}^{(3)} = v_x\,\boldsymbol{i} + v_y\,\boldsymbol{j} + v_z\,\boldsymbol{k}.$$

7.2.2. Fourier Expansion of Functions.

Remark: The ideas described above for vectors in $\mathbb{R}^3$ can be extended to functions.

Definition 2. The ***inner product*** of functions f, g on $[-L, L]$ is

$$f \cdot g = \int_{-L}^{L} f(x)\, g(x)\, dx.$$

Theorem 5. For every functions f, g, h and every a, $b \in \mathbb{R}$ holds,

(a) Positivity: $f \cdot f = 0$ iff $f = 0$; and $f \cdot f > 0$ for $f \neq 0$.

(b) Symmetry: $f \cdot g = g \cdot f$.

(c) Linearity: $(a\, f + b\, g) \cdot h = a\,(f \cdot h) + b\,(g \cdot h)$.

Remarks:

- The magnitude of a function f is

$$\|f\| = \sqrt{f \cdot f} = \Big(\int_{-L}^{L} \big(f(x)\big)^2\, dx \Big)^{1/2}.$$

- A function f is a unit function iff $\|f\| = 1$.

Definition 3. Two functions f, g are orthogonal iff $f \cdot g = 0$.

Theorem 6. An example of an orthogonal set in the space of continuous functions on $[-L, L]$ is

$$\Big\{ u_0 = \frac{1}{2}, \quad u_n = \cos\Big(\frac{n\pi x}{L}\Big), \quad v_n = \sin\Big(\frac{n\pi x}{L}\Big) \Big\}_{n=1}^{\infty}.$$

Remark: Often in the literature is used the following orthnormal set:

$$\Big\{ \tilde{u}_0 = \frac{1}{\sqrt{2L}}, \quad \tilde{u}_n = \frac{1}{\sqrt{L}} \cos\Big(\frac{n\pi x}{L}\Big), \quad \tilde{v}_n = \frac{1}{\sqrt{L}} \sin\Big(\frac{n\pi x}{L}\Big) \Big\}_{n=1}^{\infty}.$$

Remark: The orthogonality of the set above is a consequence of the following:

Theorem 7. (Orthogonality) The following relations hold for all n, $m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \begin{cases} \underline{0\qquad} & n \neq m, \\ \underline{L\qquad} & n = m \neq 0, \\ \underline{2L\qquad} & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big)\,\sin\Big(\frac{m\pi x}{L}\Big)\,dx = \begin{cases} \underline{0\qquad} & n \neq m, \\ \underline{L\qquad} & n = m, \end{cases}$$

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\sin\Big(\frac{m\pi x}{L}\Big)\,dx = \underline{0\qquad}.$$

Proof: Just recall the following trigonometric identities:

$$\cos(\theta)\,\cos(\phi) = \frac{1}{2}\big[\cos(\theta+\phi) + \cos(\theta-\phi)\big],$$

$$\sin(\theta)\,\sin(\phi) = \frac{1}{2}\big[\cos(\theta-\phi) - \cos(\theta+\phi)\big],$$

$$\sin(\theta)\,\cos(\phi) = \frac{1}{2}\big[\sin(\theta+\phi) + \sin(\theta-\phi)\big].$$

So, From the trigonometric identities above we obtain

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx + \frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

First, assume $n > 0$ or $m > 0$, then the first term vanishes, since

$$\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx = \frac{L}{2(n+m)\pi}\,\sin\Big[\frac{(n+m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

Still for $n > 0$ or $m > 0$, assume that $n \neq m$, then the second term above is

$$\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx = \frac{L}{2(n-m)\pi}\,\sin\Big[\frac{(n-m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

Again, still for $n > 0$ or $m > 0$, assume that $n = m \neq 0$, then

$$\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx = \frac{1}{2}\int_{-L}^{L} dx = L.$$

Finally, in the case that both $n = m = 0$ is simple to see that

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \int_{-L}^{L} dx = 2L.$$

The remaining equations in the theorem are proven in a similar way. $\square$

Theorem 8. (Fourier Expansion) The orthogonal set

$$\left\{u_0 = \frac{1}{2}, \quad u_n = \cos\left(\frac{n\pi x}{L}\right), \quad v_n = \sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$$

is an orthogonal basis of the space of continuous functions on $[-L, L]$, that is, any continuous function on $[-L, L]$ can be decomposed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right).$$

Moreover, the coefficients above are given by the formulas

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx,$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Furthermore, if f is piecewise continuous, then the function

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

satisfies $f_F(x) = f(x)$ for all x where f is continuous, while for all x_0 where f is discontinuous it holds

$$f_F(x_0) = \frac{1}{2}\left(\lim_{x\to x_0^+} f(x) + \lim_{x\to x_0^-} f(x)\right)$$

Example 2: Find the Fourier expansion of $f(x) = \begin{cases} \dfrac{x}{3}, & \text{for} \quad x \in [0,3] \\ 0, & \text{for} \quad x \in [-3,0). \end{cases}$

Solution: The Fourier expansion of f is

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)$$

In our case $L = 3$. We start computing b_n for $n \geqslant 1$,

$$\begin{aligned} b_n &= \frac{1}{3}\int_{-3}^{3} f(x)\sin\Big(\frac{n\pi x}{3}\Big)\,dx \\ &= \frac{1}{3}\int_{0}^{3} \frac{x}{3}\sin\Big(\frac{n\pi x}{3}\Big)\,dx \\ &= \frac{1}{9}\Big(-\frac{3x}{n\pi}\cos\Big(\frac{n\pi x}{3}\Big) + \frac{9}{n^2\pi^2}\sin\Big(\frac{n\pi x}{3}\Big)\Big)\Big|_0^3 \\ &= \frac{1}{9}\Big(-\frac{9}{n\pi}\cos(n\pi) + 0 + 0 - 0\Big) \quad \Rightarrow \quad b_n = \frac{(-1)^{(n+1)}}{n\pi}. \end{aligned}$$

A similar calculation gives us $a_n = 0$ for $n \geqslant 1$,

$$\begin{aligned} a_n &= \frac{1}{3}\int_{-3}^{3} f(x)\cos\Big(\frac{n\pi x}{3}\Big)\,dx \\ &= \frac{1}{3}\int_{0}^{3} \frac{x}{3}\cos\Big(\frac{n\pi x}{3}\Big)\,dx \\ &= \frac{1}{9}\Big(\frac{3x}{n\pi}\sin\Big(\frac{n\pi x}{3}\Big) + \frac{9}{n^2\pi^2}\cos\Big(\frac{n\pi x}{3}\Big)\Big)\Big|_0^3 \\ &= \frac{1}{9}\Big(0 + \frac{9}{n^2\pi^2}\cos(n\pi) - 0 - \frac{9}{n^2\pi^2}\Big) \quad \Rightarrow \quad a_n = \frac{((-1)^n - 1)}{n^2\pi^2}. \end{aligned}$$

Finally, we compute a_0,

$$a_0 = \frac{1}{3}\int_0^3 \frac{x}{3}\,dx = \frac{1}{9}\,\frac{x^2}{2}\Big|_0^3 = \frac{1}{2}.$$

Therefore, we get

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\Big[\frac{((-1)^n - 1)}{n^2\pi^2}\cos\Big(\frac{n\pi x}{3}\Big) + \frac{(-1)^{(n+1)}}{n\pi}\sin\Big(\frac{n\pi x}{3}\Big)\Big].$$

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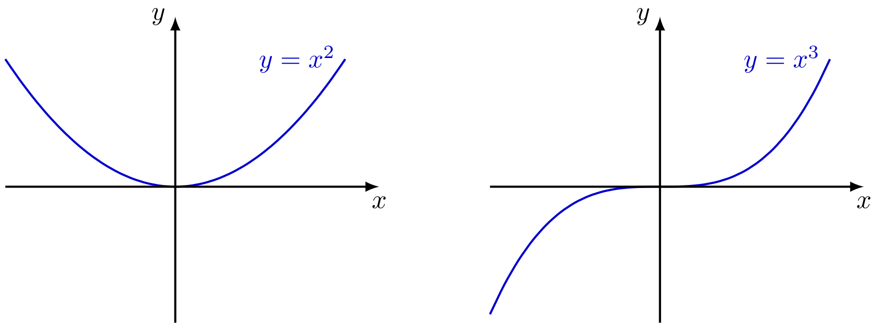
7.2.3. Odd or Even Functions.

Definition 4. A function f on $[-L, L]$ is:

- ***even*** iff $f(-x) = f(x)$ for all $x \in [-L, L]$;
- ***odd*** iff $f(-x) = -f(x)$ for all $x \in [-L, L]$.

Example 3: The function $y = x^2$ is even, while the function $y = x^3$ is odd.



Theorem 9. If f_e, g_e are even and h_o, ℓ_o are odd functions, then:

(1) $a\, f_e + b\, g_e$ is even for all a, $b \in \mathbb{R}$.

(2) $a\, h_o + b\, \ell_o$ is odd for all a, $b \in \mathbb{R}$.

(3) $f_e\, g_e$ is even.

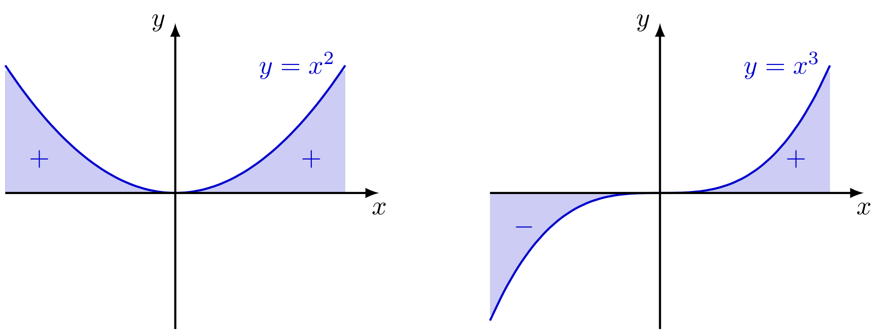
(4) $h_o\, \ell_o$ is even.

(5) $f_e\, h_o$ is odd.

(6) $\displaystyle\int_{-L}^{L} f_e\, dx = 2\int_0^L f_e\, dx$.

(7) $\displaystyle\int_{-L}^{L} h_o\, dx = 0$.

Remark:



7.2.4. Sine and Cosine Series.

Theorem 10. Let f be a function on $[-L, L]$ with a Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

(a) If the function f is <u>even</u>__________, the Fourier series above is called a <u>cosine series</u>________________, since <u>$b_n = 0$</u>__________ and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big)$$

(b) If the function f is <u>odd</u>__________, then the Fourier series above is called a <u>sine series</u>________________, since <u>$a_n = 0$</u>__________ and

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big)$$

Proof:

Part (a): Suppose that f is even, then for $n \geqslant 1$ we get

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx,$$

but f is even and the Sine is odd, so the integrand is odd. Therefore $b_n = 0$.

Part (b): Suppose that f is odd, then for $n \geqslant 1$ we get

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx,$$

but f is odd and the Cosine is even, so the integrand is odd. Therefore $a_n = 0$. Finally

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx,$$

but f is odd, hence $a_0 = 0$.

□

Example 4: Find the Fourier expansion of $f(x) = \begin{cases} 1, & \text{for} \quad x \in [0,3] \\ -1, & \text{for} \quad x \in [-3,0). \end{cases}$

Solution: The Fourier expansion of f is

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)$$

In our case $L = 3$. We start computing b_n for $n \geqslant 1$,

$$\begin{aligned} b_n &= \frac{1}{3}\int_{-3}^{3} f(x)\sin\Big(\frac{n\pi x}{3}\Big)\,dx \\ &= \frac{1}{3}\Big(\int_{-3}^{0}(-1)\sin\Big(\frac{n\pi x}{3}\Big)\,dx + \int_{0}^{3}\sin\Big(\frac{n\pi x}{3}\Big)\,dx\Big) \\ &= \frac{2}{3}\int_{0}^{3}\sin\Big(\frac{n\pi x}{3}\Big)\,dx \\ &= \frac{2}{3}\,\frac{3}{n\pi}(-1)\cos\Big(\frac{n\pi x}{3}\Big)\Big|_0^3 \\ &= \frac{2}{n\pi}\big(-(-1)^n + 1\big) \quad \Rightarrow \quad b_n = \frac{2}{n\pi}\,((-1)^{(n+1)} + 1). \end{aligned}$$

A similar calculation shows $a_n = 0$ for $n \geqslant 1$. Finally

$$a_0 = \frac{1}{3}\Big(\int_{-3}^{0} dx + \int_{0}^{3} dx\Big) = \frac{1}{3}(-3+3) = 0.$$

Therefore, we get

$$f_F(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\,((-1)^{(n+1)} + 1)\,\sin\Big(\frac{n\pi x}{L}\Big).$$

Remark: The Fourier approximation of order $N \geqslant 1$ is

$$f_N(x) = \sum_{n=1}^{N} \frac{2}{n\pi}\,((-1)^{(n+1)} + 1)\,\sin\Big(\frac{n\pi x}{L}\Big).$$

$\lhd$

Example 5 (Extra Example): Find the Fourier series expansion of the function

$$f(x) = \begin{cases} x & x \in [0,1], \\ -x & x \in [-1,0). \end{cases}$$

Solution: Since f is even, then $b_n = 0$. And since $L = 1$, we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x),$$

We start with a_0. Since f is even, a_0 is given by

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^1 x\,dx = 2\,\frac{x^2}{2}\Big|_0^1 \quad \Rightarrow \quad a_0 = 1.$$

Now we compute the a_n for $n \geqslant 1$. Since f and the cosines are even, so is their product,

$$\begin{aligned} a_n &= 2\int_0^1 x\cos(n\pi x)\,dx \\ &= 2\Big(\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{n^2\pi^2}\cos(n\pi x)\Big)\Big|_0^1 \\ &= \frac{2}{n^2\pi^2}\big(\cos(n\pi) - 1\big) \quad \Rightarrow \quad a_n = \frac{2}{n^2\pi^2}\big((-1)^n - 1\big). \end{aligned}$$

So,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big((-1)^n - 1\big)\cos(n\pi x).$$

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Example 6 (Extra Example): Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 - x & x \in [0, 1] \\ 1 + x & x \in [-1, 0). \end{cases}$$

Solution: Since f is even, then $b_n = 0$. And since $L = 1$, we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x),$$

We start computing a_0,

$$\begin{aligned} a_0 &= \int_{-1}^{1} f(x)\,dx \\ &= \int_{-1}^{0} (1 + x)\,dx + \int_{0}^{1} (1 - x)\,dx \\ &= \Big(x + \frac{x^2}{2}\Big)\Big|_{-1}^{0} + \Big(x - \frac{x^2}{2}\Big)\Big|_{0}^{1} \\ &= \Big(1 - \frac{1}{2}\Big) + \Big(1 - \frac{1}{2}\Big) \quad \Rightarrow \quad a_0 = 1. \end{aligned}$$

Similarly,

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\,\cos(n\pi x)\,dx \\ &= \int_{-1}^{0} (1 + x)\,\cos(n\pi x)\,dx + \int_{0}^{1} (1 - x)\,\cos(n\pi x)\,dx. \end{aligned}$$

Recalling the integrals

$$\begin{aligned} \int \cos(n\pi x)\,dx &= \frac{1}{n\pi}\,\sin(n\pi x), \\ \int x\,\cos(n\pi x)\,dx &= \frac{x}{n\pi}\,\sin(n\pi x) + \frac{1}{n^2\pi^2}\,\cos(n\pi x), \end{aligned}$$

it is not difficult to see that

$$\begin{aligned} a_n &= \frac{1}{n\pi}\,\sin(n\pi x)\Big|_{-1}^{0} + \Big[\frac{x}{n\pi}\,\sin(n\pi x) + \frac{1}{n^2\pi^2}\,\cos(n\pi x)\Big]\Big|_{-1}^{0} \\ &\quad + \frac{1}{n\pi}\,\sin(n\pi x)\Big|_{0}^{1} - \Big[\frac{x}{n\pi}\,\sin(n\pi x) + \frac{1}{n^2\pi^2}\,\cos(n\pi x)\Big]\Big|_{0}^{1} \\ &= \Big[\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2}\cos(-n\pi)\Big] - \Big[\frac{1}{n^2\pi^2}\cos(-n\pi) - \frac{1}{n^2\pi^2}\Big], \end{aligned}$$

we then conclude that

$$a_n = \frac{2}{n^2\pi^2}\big[1 - \cos(-n\pi)\big] = \frac{2}{n^2\pi^2}\big(1 - (-1)^n\big).$$

So,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big(1 - (-1)^n\big)\,\cos(n\pi x).$$

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7.3. The Heat Equation

Section Objective(s):

- The Heat Equation (One-Space Dim).
- The IBVP: Dirichlet Conditions.
- The IBVP: Neumann Conditions.

Remarks:

- We solve a partial differential equation: the heat equation.
- This is both a BVP and an IVP.
- We solve the heat equation using the separation of variables method.
- One first solves the BVP, which is an eigenfunction problem.
- The general solution of the BVP is a linear combination of all these eigenfunctions.
- One then uses the Fourier expansion formulas to find the unique combination of all eigenfunctions that satisfy the prescribed initial condition.
- We solve the heat equation for two types of boundary conditions: Dirichlet conditions and Neumann conditions.

7.3.1. The Heat Equation in (One-Space Dim).

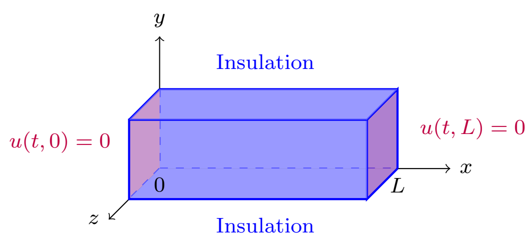
Definition 1. The ***heat*** equation in one-space dimension, for the function u depending on t and x is

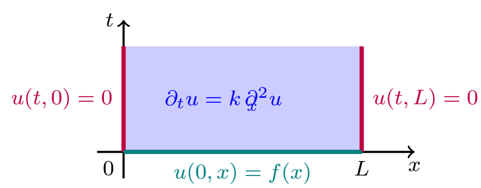
$$\partial_t u(t,x) = k\,\partial_x^2 u(t,x), \qquad \text{for} \qquad t \in [0,\infty), \qquad x \in [0,L],$$

where $k > 0$ is a constant.

Remarks:

- u is the temperature of a solid material.
- t is time, x is space.
- $k > 0$ is the heat conductivity.
- The partial differential equation above has infinitely many solutions.
- We look for solutions satisfying both:
  - Boundary conditions.
  - Initial conditions.





Boundary Conditions: $\begin{cases} u(t,0) = 0, \\ u(t,L) = 0. \end{cases}$ Initial Conditions: $\begin{cases} u(0,x) = f(x), \\ f(0) = f(L) = 0. \end{cases}$

7.3.2. The IBVP: Dirichlet Conditions.

Theorem 1 (Dirichlet). The BVP for the one-space dimensional heat equation,

$$\partial_t u = k\,\partial_x^2 u, \qquad \text{BC:} \qquad u(t,0)=0, \qquad u(t,L)=0,$$

where $k>0$, $L>0$ are constants, has infinitely many solutions

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big), \qquad c_n \in \mathbb{R}.$$

Furthermore, for every continuous function f on $[0,L]$ satisfying $f(0)=f(L)=0$, there is a unique solution u of the boundary value problem above that also satisfies the initial condition

$$u(0,x) = f(x).$$

This solution u is given by the expression above, where the coefficients c_n are

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx.$$

Remarks:

(a) This is an Initial-Boundary Value Problem (IBVP).

(b) The boundary conditions are called Dirichlet boundary conditions.

Remark: The physical meaning of the initial-boundary conditions is simple.

(1) The boundary conditions is to keep the temperature at the sides of the bar constant.

(2) The initial condition is the initial temperature on the whole bar.

Remark: The proof is based on the separation of variables method.

(1) Look for simple solutions of the BVP.

(2) Linear combination of simple solutions are solutions. (Superposition.)

(3) Determine the free constants using the initial condition.

Proof of the Theorem: First look for simple solutions of the heat equation. These simple solutions have the variables separated in two functions,

$$u(t,x) = v(t)\,w(x).$$

This separation of variables in the function also separates the heat equation,

$$\dot{v}(t)\,w(x) = k\,v(t)\,w''(x) \quad \Rightarrow \quad \frac{1}{k}\,\frac{\dot{v}(t)}{v(t)} = \frac{w''(x)}{w(x)},$$

where we used the notation $\dot{v} = dv/dt$ and $w' = dw/dx$. The only solution to the equation above is that both sides are equal the same constant, call it $-\lambda$,

$$\frac{1}{k}\,\frac{\dot{v}(t)}{v(t)} = -\lambda, \qquad \text{and} \qquad \frac{w''(x)}{w(x)} = -\lambda.$$

This separation of variables also translates to the boundary condition,

$$\left.\begin{aligned} u(t,0) &= v(t)\,w(0) = 0 \quad \text{for all } t \geqslant 0\\ u(t,L) &= v(t)\,w(L) = 0 \quad \text{for all } t \geqslant 0 \end{aligned}\right\} \quad \Rightarrow \quad w(0) = w(L) = 0.$$

Therefore, we have two solve to differential equations:

$$\dot{v}(t) = -k\lambda\,v(t), \qquad \text{and} \qquad w''(x) + \lambda\,w(x) = 0, \qquad w(0) = w(L) = 0.$$

The first equation is first order and simple to solve. The solution depends on λ,

$$v_\lambda(t) = c_\lambda\,e^{-k\lambda t}, \qquad c_\lambda = v_\lambda(0).$$

The second equation is an eigenfunction problem, which we solved in the previous section,

$$\lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \qquad w_n(x) = \sin\Big(\frac{n\pi x}{L}\Big), \qquad n = 1, 2, \cdots.$$

Since we now know the values of λ_n, we introduce them in v_λ, now called v_n,

$$v_n(t) = c_n\,e^{-k(\frac{n\pi}{L})^2 t}.$$

Therefore, we got a simple solution of the heat equation BVP,

$$u_n(t,x) = c_n\, e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big),$$

where $n = 1, 2, \cdots$. Since the boundary conditions for u_n are homogeneous, then any linear combination of the u_n is also a solution of the heat equation with homogenous boundary conditions. So the most general solution of the BVP for the heat equation is

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big).$$

Here the c_n are arbitrary constants. Now we look for the solution of the heat equation that in addition satisfies the initial condition $u(0,x) = f(x)$, where $f(0) = f(L) = 0$. This initial condition is a condition on the constants c_n, because $f(x) = u(0,x)$ is

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi x}{L}\Big).$$

The problem now is, given f, find the coefficients c_n such that the equation above holds. One way to find the c_n is to use the Fourier formulas from the previous section. These formulas apply to functions on $[-L, L]$. So, given f on $[0, L]$, we extend it to the domain $[-L, L]$ as an odd function,

$$f_{\text{odd}}(x) = f(x) \qquad \text{and} \qquad f_{\text{odd}}(-x) = -f(x), \qquad x \in [0, L]$$

Since $f(0) = 0$, we get that f_{odd} is continuous on $[-L, L]$. So f_{odd} has a Fourier series expansion. Since f_{odd} is odd, the Fourier series is a sine series

$$f_{\text{odd}}(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big)$$

and the coefficients are given by the formula

$$b_n = \frac{1}{L} \int_{-L}^{L} f_{\text{odd}}(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

Since $f_{\text{odd}}(x) = f(x)$ for $x \in [0, L]$, then $c_n = b_n$. This establishes the Theorem.

$\square$

Example 1: (Dirichlet): Find the solution to the initial-boundary value problem

$$4\,\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$$

with initial and boundary conditions given by

$$\text{IC:} \quad u(0,x) = \begin{cases} 0 & x \in \big[0, \frac{2}{3}\big), \\ 5 & x \in \big[\frac{2}{3}, \frac{4}{3}\big], \\ 0 & x \in \big(\frac{4}{3}, 2\big], \end{cases} \qquad \text{BC:} \quad \begin{cases} u(t,0) = 0, \\ u(t,2) = 0. \end{cases}$$

Solution: We look for simple solutions of the form $u(t,x) = v(t)\,w(x)$,

$$4w(x)\,\frac{dv}{dt}(t) = v(t)\,\frac{d^2w}{dx^2}(x) \quad \Rightarrow \quad \frac{4\dot{v}(t)}{v(t)} = \frac{w''(x)}{w(x)} = -\lambda.$$

So, the equations for v and w are

$$\dot{v}(t) = -\frac{\lambda}{4}\,v(t), \qquad w''(x) + \lambda\,w(x) = 0.$$

The solution for v depends on λ, and is given by

$$v_\lambda(t) = c_\lambda\,e^{-\frac{\lambda}{4}t}, \qquad c_\lambda = v_\lambda(0).$$

Next, we turn to the equation for w, and we solve the BVP

$$w''(x) + \lambda\,w(x) = 0, \quad \text{with BC} \quad w(0) = w(2) = 0.$$

This is an eigenfunction problem for w and λ. This problem has solution only for $\lambda > 0$, since only in that case the characteristic polynomial has complex roots. Let $\lambda = \mu^2$, then

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu\,i.$$

The general solution of the differential equation is

$$w_n(x) = c_1\cos(\mu x) + c_2\sin(\mu x).$$

The first boundary conditions on w implies

$$0 = w(0) = c_1, \quad \Rightarrow \quad w(x) = c_2\,\sin(\mu x).$$

The second boundary condition on w implies

$$0 = w(2) = c_2\,\sin(\mu 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_n = \Big(\frac{n\pi}{2}\Big)^2, \qquad w_n(x) = \sin\Big(\frac{n\pi x}{2}\Big), \quad n = 1, 2, \cdots.$$

Using the values of λ_n found above in the formula for v_λ we get

$$v_n(t) = c_n\,e^{-\frac{1}{4}(\frac{n\pi}{4})^2 t}, \qquad c_n = v_n(0).$$

Therefore, we get

$$u(t,x) = \sum_{n=1}^{\infty} c_n\,e^{-(\frac{n\pi}{4})^2 t}\,\sin\Big(\frac{n\pi x}{2}\Big).$$

The initial condition is

$$f(x) = u(0,x) = \begin{cases} 0 & x \in [0, \frac{2}{3}), \\ 5 & x \in [\frac{2}{3}, \frac{4}{3}], \\ 0 & x \in (\frac{4}{3}, 2]. \end{cases}$$

We extend this function to $[-2,2]$ as an odd function, so we obtain the same sine function,

$$f_{\rm odd}(x) = f(x) \quad \text{and} \quad f_{\rm odd}(-x) = -f(x), \quad \text{where} \quad x \in [0,2].$$

The Fourier expansion of $f_{\rm odd}$ on $[-2,2]$ is a sine series

$$f_{\rm odd}(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{2}\Big).$$

The coefficients b_n are given by

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\Big(\frac{n\pi x}{2}\Big)\,dx = \int_{2/3}^{4/3} 5\sin\Big(\frac{n\pi x}{2}\Big)\,dx = -\frac{10}{n\pi}\cos\Big(\frac{n\pi x}{2}\Big)\Big|_{2/3}^{4/3}.$$

So we get

$$b_n = -\frac{10}{n\pi}\Big(\cos\Big(\frac{2n\pi}{3}\Big) - \cos\Big(\frac{n\pi}{3}\Big)\Big).$$

Since $f_{\rm odd}(x) = f(x)$ for $x \in [0,2]$ we get that $c_n = b_n$. So, the solution of the initial-boundary value problem for the heat equation contains is

$$u(t,x) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\Big(\cos\Big(\frac{n\pi}{3}\Big) - \cos\Big(\frac{2n\pi}{3}\Big)\Big)\, e^{-(\frac{n\pi}{4})^2 t}\,\sin\Big(\frac{n\pi x}{2}\Big).$$

◁

7.3.3. The IBVP: Neumann Conditions.

Theorem 2 (Neumann). The BVP for the one-space dimensional heat equation,

$$\partial_t u = k\,\partial_x^2 u, \qquad \text{BC:} \qquad \partial_x u(t,0) = 0, \qquad \partial_x u(t,L) = 0,$$

where $k > 0$, $L > 0$ are constants, has infinitely many solutions

$$u(t,x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \cos\Big(\frac{n\pi x}{L}\Big), \qquad c_n \in \mathbb{R}.$$

Furthermore, for every continuous function f on $[0,L]$ satisfying $f'(0) = f'(L) = 0$, there is a unique solution u of the boundary value problem above that also satisfies the initial condition

$$u(0,x) = f(x).$$

This solution u is given by the expression above, where the coefficients c_n are

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx, \qquad n = 0, 1, 2, \cdots.$$

Remarks:

(a) This is an Initial-Boundary Value Problem (IBVP).

(b) The boundary conditions are called Neumann boundary conditions.

Remark: The physical meaning of the initial-boundary conditions is simple.

(1) The boundary conditions is to keep the heat flux at the sides of the bar constant.

(2) The initial condition is the initial temperature on the whole bar.

Remark: One can use Dirichlet conditions on one side and Neumann on the other side. This is called a mixed boundary condition.

Remark: The proof is based on the separation of variables method.

Proof of the Theorem: First look for simple solutions of the heat equation. These simple solutions have the variables separated in two functions,

$$u(t,x) = v(t)\,w(x).$$

This separation of variables in the function also separates the heat equation,

$$\dot v(t)\,w(x) = k\,v(t)\,w''(x) \quad \Rightarrow \quad \frac{1}{k}\,\frac{\dot v(t)}{v(t)} = \frac{w''(x)}{w(x)},$$

where we used the notation $\dot v = dv/dt$ and $w' = dw/dx$. The only solution to the equation above is that both sides are equal the same constant, call it $-\lambda$,

$$\frac{1}{k}\,\frac{\dot v(t)}{v(t)} = -\lambda, \qquad \text{and} \qquad \frac{w''(x)}{w(x)} = -\lambda.$$

This separation of variables also translates to the boundary condition,

$$\left.\begin{aligned} \partial_x u(t,0) = v(t)\,w'(0) = 0 \quad \text{for all } t \geqslant 0\\ \partial_x u(t,L) = v(t)\,w'(L) = 0 \quad \text{for all } t \geqslant 0 \end{aligned}\right\} \quad \Rightarrow \quad w'(0) = w'(L) = 0.$$

Therefore, we have two solve to differential equations:

$$\dot v(t) = -k\lambda\,v(t), \qquad \text{and} \qquad w''(x) + \lambda\,w(x) = 0, \qquad w'(0) = w'(L) = 0.$$

The first equation is first order and simple to solve. The solution depends on λ,

$$v_\lambda(t) = c_\lambda\,e^{-k\lambda t}, \qquad c_\lambda = v_\lambda(0).$$

The second equation is an eigenfunction problem, which has solutions only for $\lambda \geqslant 0$, since for $\lambda < 0$ the associated characteristic polynomial has real and different roots. In the case $\lambda = 0$ we get,

$$w''(x) = 0, \qquad w'(0) = w'(L) = 0 \quad \Rightarrow \quad w(x) = c \in \mathbb{R}.$$

Since any constant is solution for $\lambda = 0$, we choose the eigenfunction $w_0 = 1/2$. In the case $\lambda > 0$, we write $\lambda = \mu^2$, for $\mu > 0$, we get the general solution

$$w(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

The boundary conditions apply on the derivative,

$$w'(x) = -\mu c_1 \sin(\mu x) + \mu c_2 \cos(\mu x).$$

The boundary conditions are

$$0 = w'(0) = \mu c_2 \quad \Rightarrow \quad c_2 = 0.$$

So the function is $w(x) = \mu c_1 \cos(\mu x)$. The second boundary condition is

$$0 = w'(L) = -\mu c_1 \sin(\mu L) \quad \Rightarrow \quad \sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi, \qquad n = 1, 2, \cdots.$$

So we get the eigenvalues and eigenfunctions

$$\lambda_0 = 0, \quad w_0 = \frac{1}{2}, \quad \text{and} \quad \lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \quad w_n(x) = \cos\Big(\frac{n\pi x}{L}\Big), \quad n = 1, 2, \cdots.$$

Since we now know the values of λ_n, we introduce them in v_λ, now called v_n,

$$v_n(t) = c_n\,e^{-k(\frac{n\pi}{L})^2 t}.$$

Therefore, we got a simple solution of the heat equation BVP,

$$u_0 = \frac{1}{2}, \quad \text{and} \quad u_n(t,x) = c_n\,e^{-k(\frac{n\pi}{L})^2 t} \cos\Big(\frac{n\pi x}{L}\Big), \quad n = 1, 2, \cdots.$$

Since the boundary conditions for u_n are homogeneous, then any linear combination of the u_n is also a solution of the heat equation with homogenous boundary conditions. So the most general solution of the BVP for the heat equation is

$$u(t,x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \cos\Big(\frac{n\pi x}{L}\Big).$$

Here the c_n are arbitrary constants. Now we look for the solution of the heat equation that in addition satisfies the initial condition $u(0,x) = f(x)$, where $f'(0) = f'(L) = 0$. This initial condition is a condition on the constants c_n, because $f(x) = u(0,x)$ is

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\Big(\frac{n\pi x}{L}\Big).$$

The problem now is, given f, find the coefficients c_n such that the equation above holds. One way to find the c_n is to use the Fourier formulas from the previous section. These formulas apply to functions on $[-L, L]$. So, given f on $[0, L]$, we extend it to the domain $[-L, L]$ as an even function,

$$f_{\text{even}}(x) = f(x) \qquad \text{and} \qquad f_{\text{even}}(-x) = f(x), \qquad x \in [0, L]$$

We get that f_{even} is continuous on $[-L, L]$. So f_{even} has a Fourier series expansion. Since f_{even} is even, the Fourier series is a cosine series

$$f_{\text{even}}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big)$$

and the coefficients are given by the formula

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{\text{even}}(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L}\int_0^L f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx, \qquad n = 0, 1, 2, \cdots.$$

Since $f_{\text{even}}(x) = f(x)$ for $x \in [0, L]$, then $c_n = a_n$ for $n = 0, 1, 2, \cdots$. This establishes the Theorem.

$\square$

Example 2: (Neumann): Find the solution to the initial-boundary value problem

$$\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 3],$$

with initial and boundary conditions given by

$$\text{IC:} \quad u(0, x) = \begin{cases} 7 & x \in \big[\frac{3}{2}, 3\big], \\ 0 & x \in \big[0, \frac{3}{2}\big), \end{cases} \qquad \text{BC:} \quad \begin{cases} u'(t, 0) = 0, \\ u'(t, 3) = 0. \end{cases}$$

Solution: We look for simple solutions of the form $u(t, x) = v(t)\, w(x)$,

$$w(x)\, \frac{dv}{dt}(t) = v(t)\, \frac{d^2 w}{dx^2}(x) \quad \Rightarrow \quad \frac{\dot{v}(t)}{v(t)} = \frac{w''(x)}{w(x)} = -\lambda.$$

This separation of variables also translates to the boundary condition,

$$\left.\begin{aligned} \partial_x u(t, 0) = v(t)\, w'(0) = 0 \quad \text{for all } t \geqslant 0 \\ \partial_x u(t, 3) = v(t)\, w'(3) = 0 \quad \text{for all } t \geqslant 0 \end{aligned}\right\} \quad \Rightarrow \quad w'(0) = w'(3) = 0.$$

So, the equations for v and w are

$$\dot{v}(t) = -\lambda\, v(t), \qquad \text{and} \qquad w''(x) + \lambda\, w(x) = 0 \qquad w'(0) = w'(3) = 0.$$

The solution for v depends on λ, and is given by

$$v_\lambda(t) = c_\lambda\, e^{-\lambda t}, \qquad c_\lambda = v_\lambda(0).$$

The equation for w is an eigenfunction problem that has solution for $\lambda \geqslant 0$, since for $\lambda < 0$ the associated characteristic polynomial has real and different roots. In the case $\lambda = 0$,

$$w''(x) = 0, \qquad w'(0) = w'(3) = 0 \quad \Rightarrow \quad w(x) = c \in \mathbb{R}.$$

Since any constant is solution for $\lambda = 0$, we choose the eigenfunction $w_0 = 1/2$. In the case $\lambda > 0$, we write $\lambda = \mu^2$, for $\mu > 0$,

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm \mu\, i.$$

The general solution of the differential equation is

$$w_n(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Its derivative is

$$w'(x) = -\mu\, c_1 \sin(\mu x) + \mu\, c_2 \cos(\mu x).$$

The first boundary conditions on w implies

$$0 = w'(0) = \mu\, c_2, \quad \Rightarrow \quad c_2 = 0 \quad \Rightarrow \quad w(x) = c_1 \cos(\mu x).$$

The second boundary condition on w implies

$$0 = w'(3) = -\mu\, c_1 \sin(\mu 3), \quad c_1 \neq 0, \quad \Rightarrow \quad \sin(\mu 3) = 0.$$

Then, $\mu_n 3 = n\pi$, for $n = 1, 2, \cdots$. That is, $\mu_n = \dfrac{n\pi}{3}$. Choosing $c_2 = 1$, and recaling the case $\lambda = 0$,

$$\lambda_0 = 0, \quad w_0 = \frac{1}{2}, \quad \text{and} \quad \lambda_n = \Big(\frac{n\pi}{3}\Big)^2, \quad w_n(x) = \cos\Big(\frac{n\pi x}{3}\Big), \quad n = 1, 2, \cdots.$$

Using the values of λ_n found above in the formula for v_λ we get

$$v_n(t) = c_n\, e^{-(\frac{n\pi}{3})^2 t}, \qquad c_n = v_n(0), \qquad n = 0, 1, 2, \cdots.$$

Therefore, we get

$$u(t,x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{3})^2 t} \cos\Big(\frac{n\pi x}{2}\Big).$$

The initial condition is

$$f(x) = u(0,x) = \begin{cases} 7 & x \in [\frac{3}{2}, 3], \\ 0 & x \in [0, \frac{3}{2}), \end{cases}$$

We extend f to $[-3,3]$ as an even function

$$f_{\text{even}}(x) = \begin{cases} 7 & x \in [\frac{3}{2}, 3], \\ 0 & x \in [-\frac{3}{2}, \frac{3}{2}), \\ 7 & x \in [-3, -\frac{3}{2}]. \end{cases}$$

Since f_{even} is even, its Fourier expansion is a cosine series

$$f_{\text{even}}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{3}\Big).$$

The coefficient a_0 is given by

$$a_0 = \frac{2}{3} \int_0^3 f(x)\, dx = \frac{2}{3} \int_{3/2}^3 7\, dx = \frac{2}{3}\, 7\, \frac{3}{2} \quad \Rightarrow \quad a_0 = 7.$$

Now the coefficients a_n for $n \geqslant 1$ are given by

$$\begin{aligned} a_n &= \frac{2}{3} \int_0^3 f(x) \cos\Big(\frac{n\pi x}{3}\Big)\, dx = \frac{2}{3} \int_{3/2}^3 7 \cos\Big(\frac{n\pi x}{3}\Big)\, dx \\ &= \frac{2}{3}\, 7\, \frac{3}{n\pi}\, \sin\Big(\frac{n\pi x}{3}\Big)\Big|_{3/2}^{3} = \frac{2}{3}\, 7\, \frac{3}{n\pi}\, \Big(0 - \sin\Big(\frac{n\pi}{2}\Big)\Big) \\ a_n &= -\frac{14}{n\pi}\, \sin(n\pi). \end{aligned}$$

But for $n = 2k$ we have that $\sin(2k\pi/2) = \sin(k\pi) = 0$, while for $n = 2k-1$ we have that $\sin((2k-1)\pi/2) = (-1)^{k-1}$. Therefore

$$a_{2k} = 0, \qquad a_{2k-1} = \frac{14(-1)^k}{(2k-1)\pi}, k = 1, 2, \cdots.$$

We then obtain the Fourier series expansion of f_{even},

$$f_{\text{even}}(x) = \frac{7}{2} + \sum_{k=1}^{\infty} \frac{14(-1)^k}{(2k-1)\pi} \cos\Big(\frac{(2k-1)\pi x}{3}\Big)$$

But the function f has exactly the same Fourier expansion on $[0,3]$, which means that

$$c_0 = 7, \qquad c_{2k} = 0, \qquad c_{(2k-1)} = \frac{14(-1)^k}{(2k-1)\pi}.$$

So the solution of the initial-boundary value problem for the heat equation is

$$u(t,x) = \frac{7}{2} + \sum_{k=1}^{\infty} \frac{14(-1)^k}{(2k-1)\pi}\, e^{-(\frac{(2k-1)\pi}{3})^2 t} \cos\Big(\frac{(2k-1)\pi x}{3}\Big).$$

$\triangleleft$