Sine and Cosine Series (Sect. 7.2)

- Even and Odd Functions
- Main Properties of Even and Odd Functions
- Cosine and Sine Series
- Even Periodic and Odd Periodic Extensions of Functions

Even and Odd Functions

Definition

A function $f : [-L, L] \to \mathbb{R}$ is *even* iff for all $x \in [-L, L]$ holds

$$f(-x) = f(x).$$

A function $f : [-L, L] \to \mathbb{R}$ is *odd* iff for all $x \in [-L, L]$ holds

$$f(-x) = -f(x).$$

Remarks:

- The only function that is both odd and even is $f = 0$.
- Most functions are neither odd nor even.

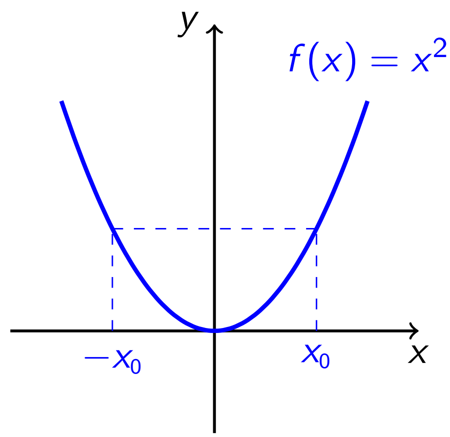
Even and Odd Functions

Example

Show that the function $f(x) = x^2$ is even on $[-L, L]$.

Solution: The function is even, since

$$f(-x) = (-x)^2 = x^2 = f(x).$$



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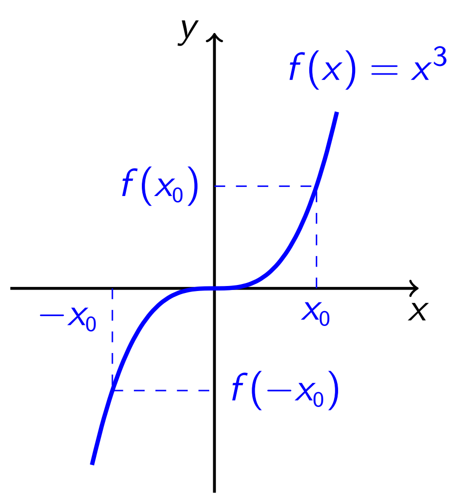
Even, odd functions.

Example

Show that the function $f(x) = x^3$ is odd on $[-L, L]$.

Solution: The function is odd, since

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$



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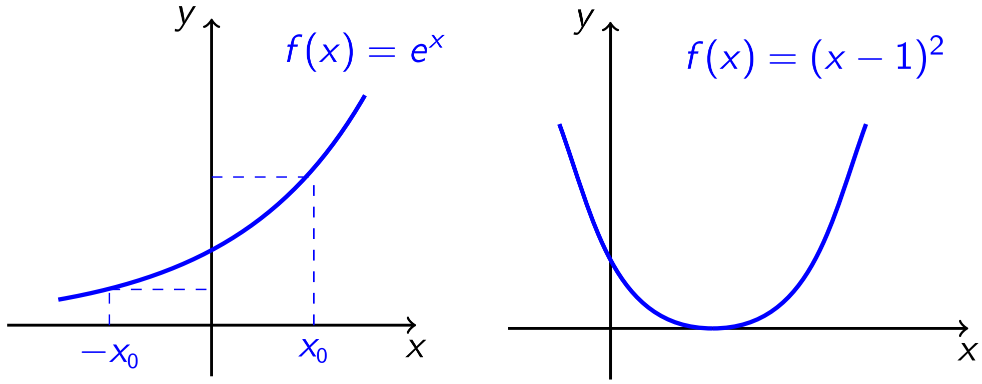
Even and Odd Functions

Example

(1) The function $f(x) = \cos(ax)$ is even on $[-L, L]$;

(2) The function $f(x) = \sin(ax)$ is odd on $[-L, L]$;

(3) The functions $f(x) = e^x$ and $f(x) = (x-2)^2$ are neither even nor odd.



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Main Properties of Even and Odd Functions

Theorem

(1) *A linear combination of even (odd) functions is even (odd).*

(2) *The product of two odd functions is even.*

(3) *The product of two even functions is even.*

(4) *The product of an even function by an odd function is odd.*

Proof:

(1) Let f and g be even, that is, $f(-x) = f(x)$, $g(-x) = g(x)$. Then, for all $a, b \in \mathbb{R}$ holds,

$$(af + bg)(-x) = af(-x) + bg(-x) = af(x) + bg(x) = (af + bg)(x).$$

Case "odd" is similar.

Main Properties of Even and Odd Functions

Theorem

(1) *A linear combination of even (odd) functions is even (odd).*

(2) *The product of two odd functions is even.*

(3) *The product of two even functions is even.*

(4) *The product of an even function by an odd function is odd.*

Proof:

(2) Let f and g be odd, that is, $f(-x) = -f(x)$, $g(-x) = -g(x)$. Then, for all $a, b \in \mathbb{R}$ holds,

$$(fg)(-x) = f(-x)g(-x) = \big(-f(x)\big)\big(-g(x)\big) = f(x)g(x) = (fg)(x).$$

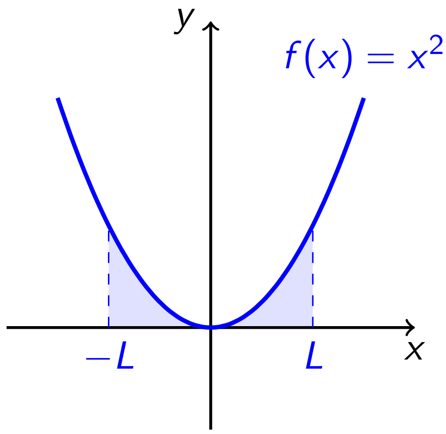
Cases (3), (4) are similar. $\square$

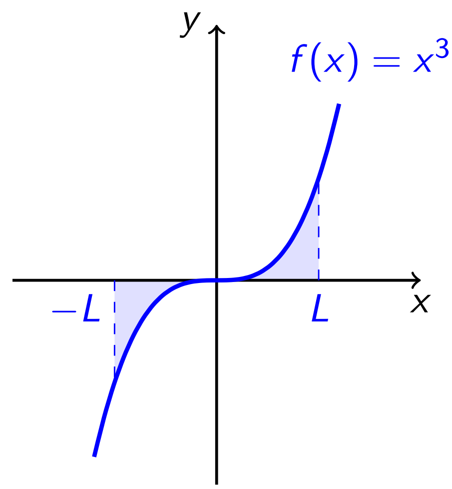
Main Properties of Even and Odd Functions

Theorem

If $f : [-L, L] \to \mathbb{R}$ is even, then $\displaystyle\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$

If $f : [-L, L] \to \mathbb{R}$ is odd, then $\displaystyle\int_{-L}^{L} f(x)\,dx = 0.$





Main Properties of Even and Odd Functions

Proof:

$$I = \int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx \quad y = -x,\ dy = -dx.$$

$$I = \int_{L}^{0} f(-y)\,(-dy) + \int_{0}^{L} f(x)\,dx = \int_{0}^{L} f(-y)\,dy + \int_{0}^{L} f(x)\,dx.$$

Even case: $f(-y) = f(y)$, therefore,

$$I = \int_{0}^{L} f(y)\,dy + \int_{0}^{L} f(x)\,dx \;\Rightarrow\; \int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

Odd case: $f(-y) = -f(y)$, therefore,

$$I = -\int_{0}^{L} f(y)\,dy + \int_{0}^{L} f(x)\,dx \quad\Rightarrow\quad \int_{-L}^{L} f(x)\,dx = 0. \qquad \square$$

Sine and Cosine Series (Sect. 7.2)

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- **Cosine and Sine Series**
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Cosine and Sine Series

Theorem (Cosine and Sine Series)

Consider the function $f : [-L, L] \to \mathbb{R}$ *with Fourier expansion*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

(1) *If* f *is even, then* $b_n = 0$ *for* $n = 1, 2, \cdots$, *and the Fourier series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big)$$

is called a Cosine Series.

(2) *If* f *is odd, then* $a_n = 0$ *for* $n = 0, 1, \cdots$, *and the Fourier series*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big)$$

is called a Sine Series.

Cosine and Sine Series

Proof:
If f is even, and since the Sine function is odd, then

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = 0,$$

since we are integrating an odd function on $[-L, L]$.

If f is odd, and since the Cosine function is even, then

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx = 0,$$

since we are integrating an odd function on $[-L, L]$. $\square$

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- **Even Periodic and Odd Periodic Extensions of Functions**

Even Periodic and Odd Periodic Extensions of Functions

(1) Even-periodic case:

A function $f : [0, L] \to \mathbb{R}$ can be extended as an even function $f : [-L, L] \to \mathbb{R}$ requiring for $x \in [0, L]$ that

$$f(-x) = f(x).$$

This function $f : [-L, L] \to \mathbb{R}$ can be further extended as a periodic function $f : \mathbb{R} \to \mathbb{R}$ requiring for $x \in [-L, L]$ that

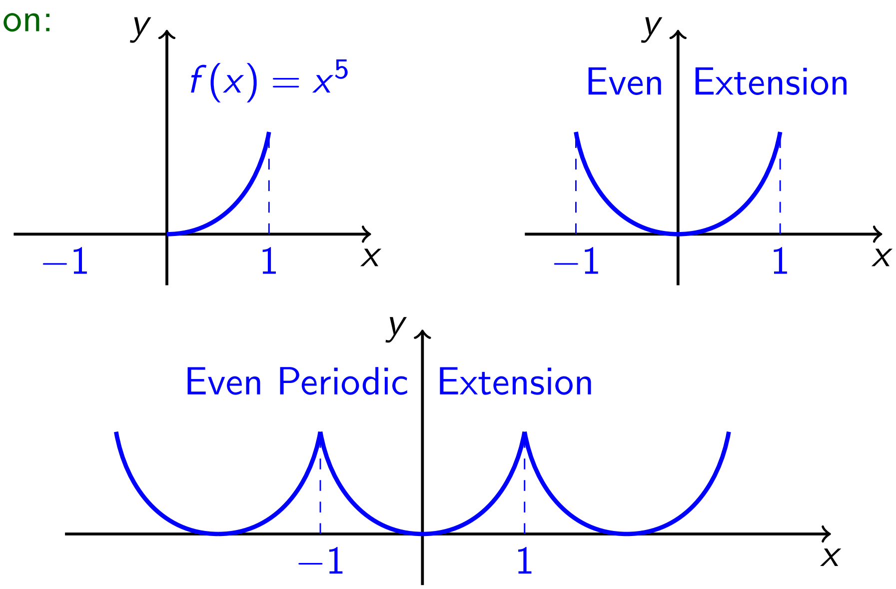
$$f(x + 2nL) = f(x).$$

Even Periodic and Odd Periodic Extensions of Functions

Example

Sketch the graph of the even-periodic extension of $f(x) = x^5$, with $x \in [0, 1]$.

Solution:



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Even Periodic and Odd Periodic Extensions of Functions

(2) Odd-periodic case:

A function $f : (0, L) \to \mathbb{R}$ can be extended as an odd function $f : (-L, L) \to \mathbb{R}$ requiring for $x \in (0, L)$ that

$$f(-x) = -f(x), \qquad f(0) = 0.$$

This function $f : (-L, L) \to \mathbb{R}$ can be further extended as a periodic function $f : \mathbb{R} \to \mathbb{R}$ requiring for $x \in (-L, L)$ and n integer that

$$f(x + 2nL) = f(x), \quad \text{and} \quad f(nL) = 0.$$

Remark: At $x = \pm L$, the extension f must satisfy:

(a) f is odd, hence $f(-L) = -f(L)$;

(b) f is periodic, hence $f(-L) = f(-L + 2L) = f(L)$.

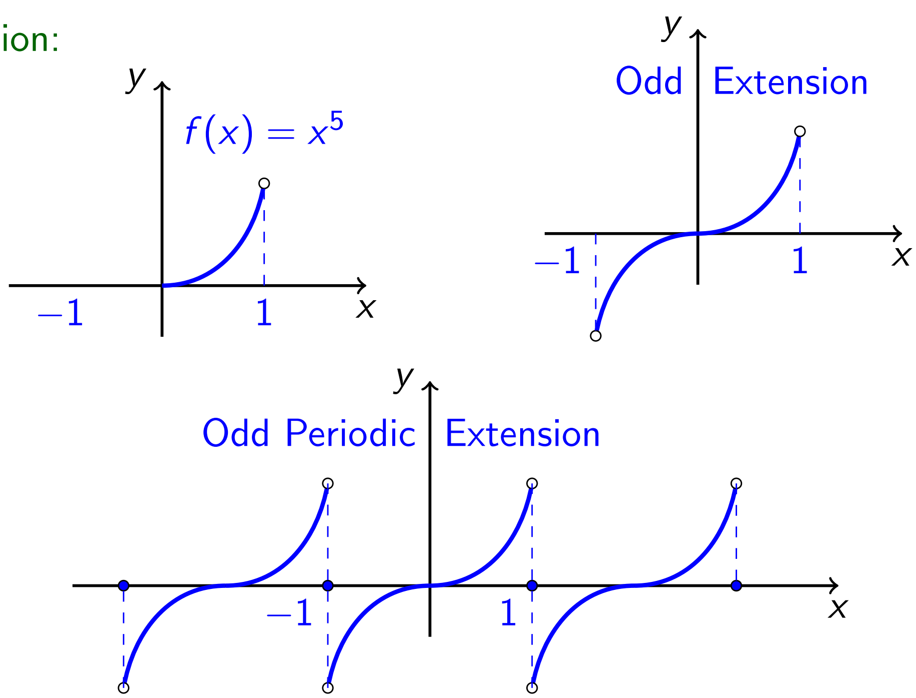
We then conclude that $-f(L) = f(L)$, hence $f(L) = 0$.

Even Periodic and Odd Periodic Extensions of Functions

Example

Graph of the odd-periodic extension of $f(x) = x^5$, with $x \in (0, 1)$.

Solution:



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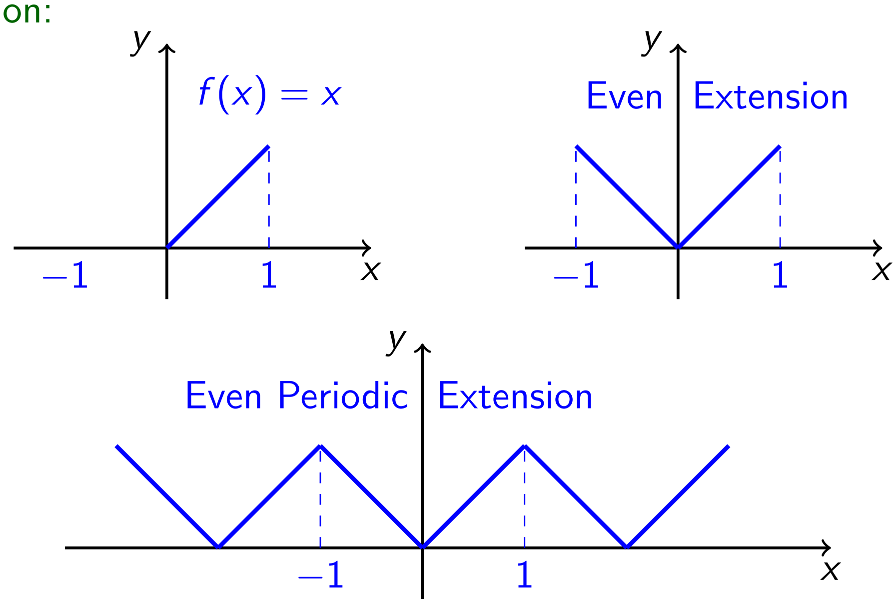
Even Periodic and Odd Periodic Extensions of Functions

Example

Sketch the graph of the even-periodic extension of $f(x) = x$, with $x \in [0, 1]$, and then find its Fourier Series.

Solution:



Even Periodic and Odd Periodic Extensions of Functions

Example

Sketch the graph of the even-periodic extension of $f(x) = x$, with $x \in [0, 1]$, and then find its Fourier Series.

Solution: Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$. From the graph: $a_0 = 1$.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L}\int_0^L f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx.$$

$$a_n = 2\int_0^1 x\cos(n\pi x)\,dx = 2\Big[\frac{x\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{(n\pi)^2}\Big]\Big|_0^1,$$

$$a_n = \frac{2}{(n\pi)^2}\big[\cos(n\pi) - 1\big] \quad \Rightarrow \quad a_n = \frac{2}{(n\pi)^2}\big[(-1)^n - 1\big].$$

Even Periodic and Odd Periodic Extensions of Functions

Example

Sketch the graph of the even-periodic extension of $f(x) = x$, with $x \in [0, 1]$, and then find its Fourier Series.

Solution: Recall: $b_n = 0$, and $a_n = \frac{2}{(n\pi)^2}\big[(-1)^n - 1\big]$.

$$n = 2k \quad \Rightarrow \quad a_{2k} = \frac{2}{[(2k)\pi]^2}\big[(-1)^{2k} - 1\big] \quad \Rightarrow \quad a_{2k} = 0.$$

$$n = 2k - 1 \;\Rightarrow\; a_{2k-1} = \frac{2\,[-1-1]}{[(2k-1)\pi]^2} \;\Rightarrow\; a_{2k-1} = \frac{-4}{[(2k-1)\pi]^2}.$$

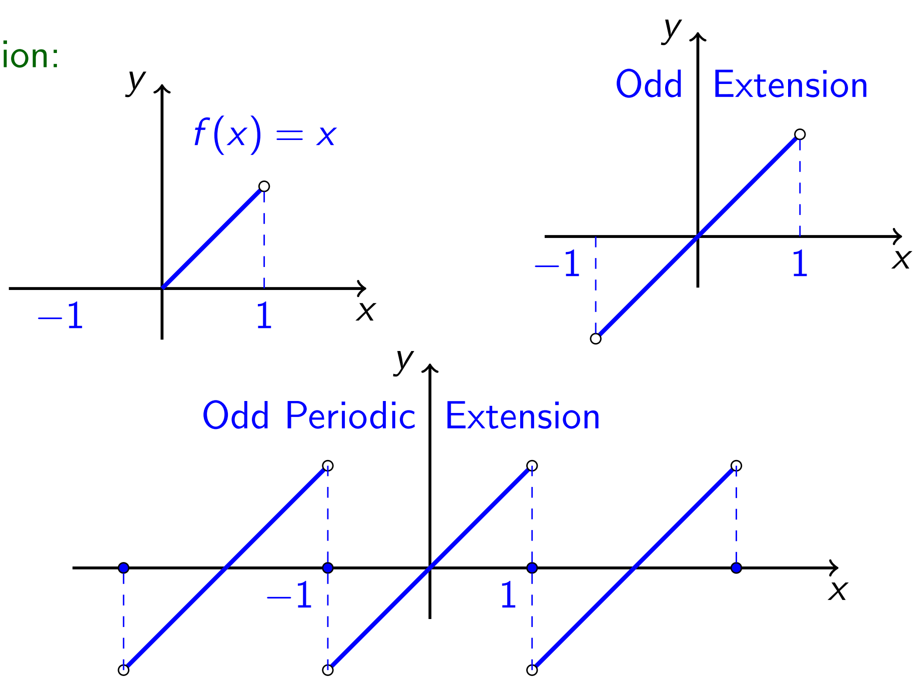
$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\big((2k-1)\pi x\big). \qquad \lhd$$

Even Periodic and Odd Periodic Extensions of Functions

Example

Sketch the graph of the odd-periodic extension of $f(x) = x$, with $x \in (0, 1)$, and then find its Fourier Series.

Solution:



Even Periodic and Odd Periodic Extensions of Functions

Example

Sketch the graph of the odd-periodic extension of $f(x) = x$, with $x \in (0,1)$, and then find its Fourier Series.

Solution: Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx.$$

$$b_n = 2\int_0^1 x\sin(n\pi x)\,dx = 2\Big[-\frac{x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{(n\pi)^2}\Big]\Big|_0^1,$$

$$b_n = \frac{-2}{n\pi}\big[\cos(n\pi) - 0\big] \quad \Rightarrow \quad b_n = \frac{-2\,(-1)^n}{n\pi}.$$

Even Periodic and Odd Periodic Extensions of Functions

Example

Sketch the graph of the odd-periodic extension of $f(x) = x$, with $x \in (0,1)$, and then find its Fourier Series.

Solution: Recall: $a_n = 0$, and $b_n = \dfrac{2\,(-1)^{n+1}}{n\pi}$. Therefore,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n}\,\sin(n\pi x). \qquad \triangleleft$$