Real, repeated eigenvalues (Sect. 5.9)

- Review: Classification of 2×2 diagonalizable systems.
- Review: The case of diagonalizable matrices.
- The algebraic multiplicity of an eigenvalue.
- Non-diagonalizable matrices with a repeated eigenvalue.
- Phase portraits for 2×2 systems.

Review: Classification of 2×2 diagonalizable systems.

Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1 = \overline{\mathbf{v}}_2$, (Section 5.8).

(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 5.9).

Remark:

(c-2) $\lambda_1 = \lambda_2$ real-valued with only one eigen-direction. Hence, A is not diagonalizable, (Section 5.9).

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Review: The case of diagonalizable matrices.

Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to the homogeneous, constant coefficients, linear system

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

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The algebraic multiplicity of an eigenvalue.

Definition

Let $\{\lambda_1, \cdots, \lambda_k\}$ be the set of eigenvalues of an $n \times n$ matrix, where $1 \leqslant k \leqslant n$, hence the characteristic polynomial is

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_k)^{r_k}.$$

The positive integer r_i, for $i = 1, \cdots, k$, is called the *algebraic multiplicity* of the eigenvalue λ_i. The eigenvalue λ_i is called *repeated* iff $r_i > 1$.

Remark:

- A matrix with repeated eigenvalues may or may not be diagonalizable.
- Equivalently: An $n \times n$ matrix with repeated eigenvalues may or may not have a linearly independent set of n eigenvectors.

The algebraic multiplicity of an eigenvalue.

Example

Show that matrix A is diagonalizable but matrix B is not, where

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: The eigenvalues of A are the solutions of

$$\begin{vmatrix} (3-\lambda) & 0 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = -(\lambda - 3)^2 (\lambda - 1) = 0,$$

We conclude: $\lambda_1 = 3$, $r_1 = 2$, and $\lambda_2 = 1$, $r_2 = 1$.

Verify that the eigenvalues are: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}$.

We conclude: A is diagonalizable.

The algebraic multiplicity of an eigenvalue.

Example

Show that matrix A is diagonalizable but matrix B is not, where

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: The eigenvalues of B are the solutions of

$$\begin{vmatrix} (3-\lambda) & 1 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = -(\lambda - 3)^2 (\lambda - 1) = 0,$$

We conclude: $\lambda_1 = 3$, $r_1 = 2$, and $\lambda_2 = 1$, $r_2 = 1$.

Verify that the eigenvalues are: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

We conclude: B is not diagonalizable.

The algebraic multiplicity of an eigenvalue.

Example

Find a fundamental set of solutions to

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

Solution: Since matrix A is diagonalizable, with eigen-pairs,

$$\lambda_1 = 3, \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad \lambda_2 = 1, \quad \left\{ \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}.$$

We conclude that a set of fundamental solutions is

$$\left\{ \mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t},\ \mathbf{x}_2(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{3t},\ \mathbf{x}_3(t) = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} e^{t} \right\}. \quad \triangleleft$$

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Non-diagonalizable matrices with a repeated eigenvalue.

Theorem (Repeated eigenvalue)

If λ is an eigenvalue of an $n \times n$ matrix A having algebraic multiplicity $r = 2$ and only one associated eigen-direction, then the differential equation

$$\mathbf{x}'(t) = A\,\mathbf{x}(t),$$

has a linearly independent set of solutions given by

$$\left\{\mathbf{x}^{(1)}(t) = \mathbf{v}\; e^{\lambda t}, \quad \mathbf{x}^{(2)}(t) = \left(\mathbf{v}\; t + \mathbf{w}\right) e^{\lambda t}\right\}.$$

where the vector $\mathbf{w}$ is solution of

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$

which always has a solution $\mathbf{w}$.

Non-diagonalizable matrices with a repeated eigenvalue.

Recall: The case of a single second order equation

$$y'' + a_1\, y' + a_0\, y = 0$$

with characteristic polynomial

$$p(r) = r^2 + a_1\, r + a_0 = (r - r_1)^2.$$

In this case a fundamental set of solutions is

$$\left\{y_1(t) = e^{r_1 t}, \quad y_2(t) = t\, e^{r_1 t}\right\}.$$

This is not the case with systems of first order linear equations,

$$\left\{\mathbf{x}^{(1)}(t) = \mathbf{v}\; e^{\lambda t}, \quad \mathbf{x}^{(2)}(t) = \left(\mathbf{v}\; t + \mathbf{w}\right) e^{\lambda t}\right\}.$$

In general, $\mathbf{w} \neq \mathbf{0}$.

Non-diagonalizable matrices with a repeated eigenvalue.

Example

Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Find the eigenvalues of A. Its characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} \left(-\frac{3}{2} - \lambda\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} - \lambda\right) \end{vmatrix} = \left(\lambda + \frac{3}{2}\right)\left(\lambda + \frac{1}{2}\right) + \frac{1}{4}.$$

So $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$. The roots and multiplicity are

$$\lambda = -1, \qquad r = 2.$$

The corresponding eigenvectors are the solutions of $(A + I)\mathbf{v} = \mathbf{0}$,

$$\begin{bmatrix} \left(-\frac{3}{2} + 1\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} + 1\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Non-diagonalizable matrices with a repeated eigenvalue.

Example

Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Recall: $\lambda = -1$, with $r = 2$, and $(A + I) \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

The eigenvector components satisfy: $v_1 = 2v_2$. We obtain,

$$\lambda = -1, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_2.$$

We conclude that this eigenvalue has only one eigen-direction.
Matrix A is not diagonalizable.
Theorem above says we need to find $\mathbf{w}$ solution of $(A + I)\mathbf{w} = \mathbf{v}$.

$$\left[\begin{array}{cc|c} -\frac{1}{2} & 1 & 2 \\ -\frac{1}{4} & \frac{1}{2} & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 1 & -2 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

Non-diagonalizable matrices with a repeated eigenvalue.

Example

Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Recall that:

$\lambda = -1$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_2$, and $(A+I)\mathbf{w} = \mathbf{v} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 0 & 0 \end{array}\right]$.

We obtain $w_1 = 2w_2 - 4$. That is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

Given a solution $\mathbf{w}$, then $c\mathbf{v} + \mathbf{w}$ is also a solution, $c \in \mathbb{R}$.

We choose the simplest solution, $\mathbf{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$. We conclude,

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}, \qquad \mathbf{x}^{(2)}(t) = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) e^{-t}. \qquad \triangleleft$$

Non-diagonalizable matrices with a repeated eigenvalue.

Example

Find the solution $\mathbf{x}$ to the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution: The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) e^{-t}.$$

The initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix}.$$

We conclude: $\mathbf{x}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{4}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) e^{-t}.$ $\triangleleft$

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Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of

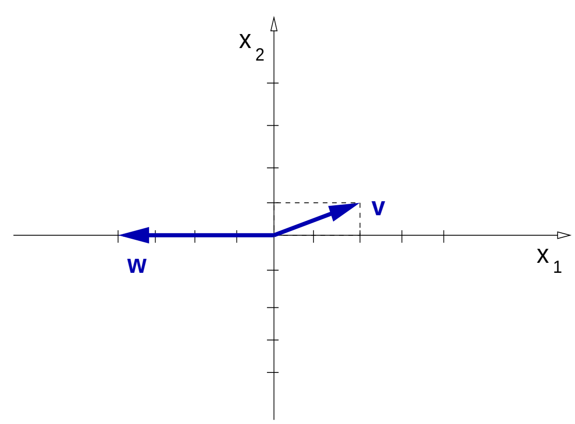
$\mathbf{x}' = A\mathbf{x}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$

Solution:

We start plotting the vectors

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\mathbf{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$$



Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of

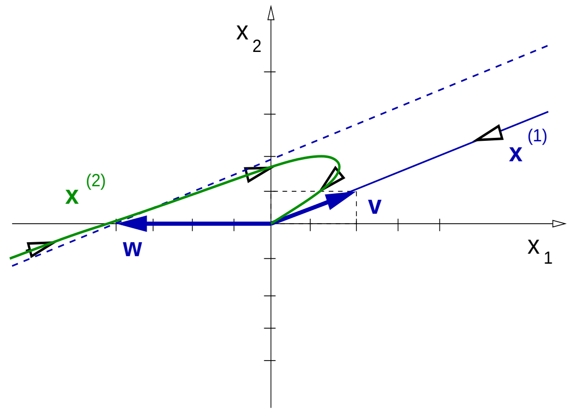
$\mathbf{x}' = A\mathbf{x}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$

Solution:

Now plot the solutions

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$$

$$\mathbf{x}^{(2)} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$$



Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of

$\mathbf{x}' = A\mathbf{x}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$

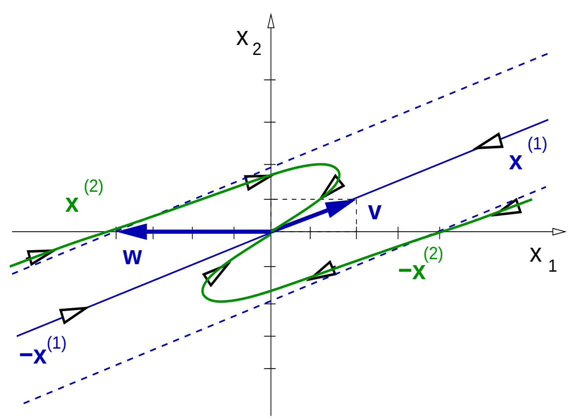
Solution:

Now plot the solutions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

$$\mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)},$$

This is the case $\lambda < 0$.



Phase portraits for 2×2 systems.

Example

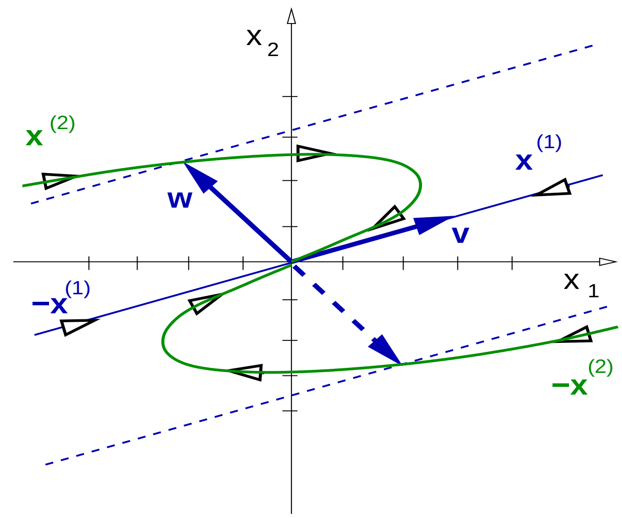
Given any vectors $\mathbf{v}$ and $\mathbf{w}$, and any constant λ, plot the phase portraits of the functions

$$\mathbf{x}^{(1)}(t) = \mathbf{v}\, e^{\lambda t}, \qquad \mathbf{x}^{(2)}(t) = \big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{\lambda t},$$

Solution:
The case $\lambda < 0$. We plot the functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

$$\mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)}.$$



Phase portraits for 2×2 systems.

Example

Given any vectors $\mathbf{v}$ and $\mathbf{w}$, and any constant λ, plot the phase portraits of the functions

$$\mathbf{x}^{(1)}(t) = \mathbf{v}\, e^{\lambda t}, \qquad \mathbf{x}^{(2)}(t) = \big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{\lambda t},$$

Solution:
The case $\lambda > 0$. We plot the functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

$$\mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)}.$$