Overview of Fourier Series (Sect. 6.2).

- Periodic functions.
- Orthogonality of Sines and Cosines.
- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.

Periodic functions.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* iff there exists $\tau > 0$ such that for all $x \in \mathbb{R}$ holds

$$f(x + \tau) = f(x).$$

Remark: f is invariant under translations by τ.

Definition

A *period* T of a periodic function f is the smallest value of τ such that $f(x + \tau) = f(x)$ holds.

Notation:
A periodic function with period T is also called T-periodic.

Periodic functions.

Example

The following functions are periodic, with period T,

$$f(x) = \sin(x), \qquad T = 2\pi.$$

$$f(x) = \cos(x), \qquad T = 2\pi.$$

$$f(x) = \tan(x), \qquad T = \pi.$$

$$f(x) = \sin(ax), \qquad T = \frac{2\pi}{a}.$$

The proof of the latter statement is the following:

$$f\Big(x + \frac{2\pi}{a}\Big) = \sin\Big(ax + a\frac{2\pi}{a}\Big) = \sin(ax + 2\pi) = \sin(ax) = f(x).$$

$\triangleleft$

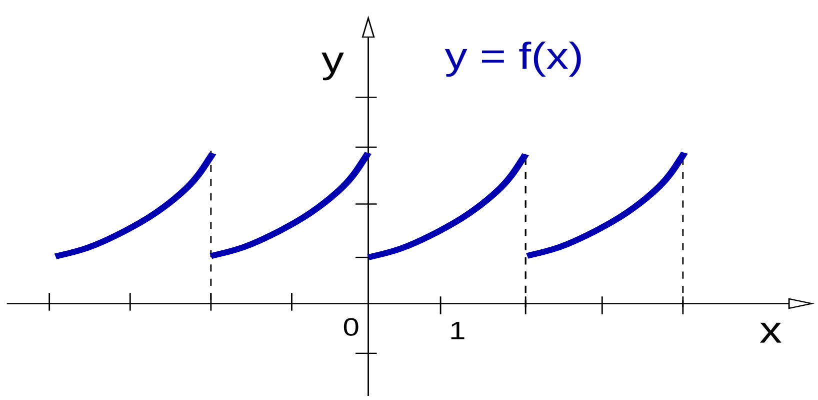
Periodic functions.

Example

Show that the function below is periodic, and find its period,

$$f(x) = e^x, \qquad x \in [0, 2), \qquad f(x-2) = f(x).$$

Solution: We just graph the function,



So the function is periodic with period $T = 2$. $\triangleleft$

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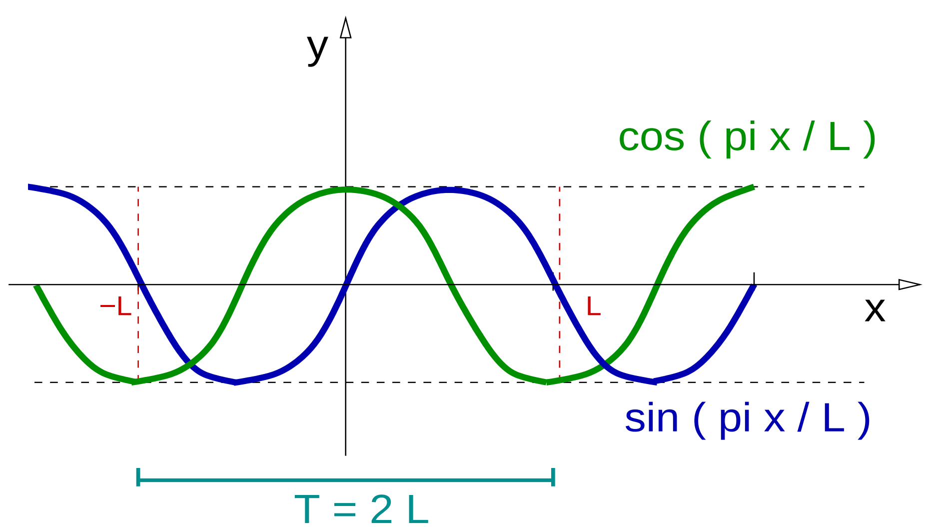
Orthogonality of Sines and Cosines.

Remark:
From now on we work on the following domain: $[-L, L]$.



Orthogonality of Sines and Cosines.

Theorem (Orthogonality)

The following relations hold for all n, $m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big)\,\sin\Big(\frac{m\pi x}{L}\Big)\,dx = \begin{cases} 0 & n \neq m, \\ L & n = m, \end{cases}$$

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\sin\Big(\frac{m\pi x}{L}\Big)\,dx = 0.$$

Remark:

- The operation $f \cdot g = \int_{-L}^{L} f(x)\,g(x)\,dx$ is an inner product in the vector space of functions. Like the dot product is in $\mathbb{R}^2$.
- Two functions f, g, are orthogonal iff $f \cdot g = 0$.

Orthogonality of Sines and Cosines.

Recall:
$$\cos(\theta)\,\cos(\phi) = \frac{1}{2}\big[\cos(\theta+\phi) + \cos(\theta-\phi)\big];$$
$$\sin(\theta)\,\sin(\phi) = \frac{1}{2}\big[\cos(\theta-\phi) - \cos(\theta+\phi)\big];$$
$$\sin(\theta)\,\cos(\phi) = \frac{1}{2}\big[\sin(\theta+\phi) + \sin(\theta-\phi)\big].$$

Proof: First formula: If $n = m = 0$, it is simple to see that

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \int_{-L}^{L} dx = 2L.$$

In the case where one of n or m is non-zero, use the relation

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx + \frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

Orthogonality of Sines and Cosines.

Proof: Since one of n or m is non-zero, holds

$$\frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx = \frac{L}{2(n+m)\pi}\,\sin\Big[\frac{(n+m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

We obtain that

$$\int_{-L}^{L}\cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

If we further restrict $n \neq m$, then

$$\frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx = \frac{L}{2(n-m)\pi}\,\sin\Big[\frac{(n-m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

If $n = m \neq 0$, we have that

$$\frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx = \frac{1}{2}\int_{-L}^{L}dx = L.$$

This establishes the first equation in the Theorem. The remaining equations are proven in a similar way. □

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The Fourier Theorem: Continuous case.

Theorem (Fourier Series)

If the function $f : [-L, L] \subset \mathbb{R} \to \mathbb{R}$ is continuous, then f can be expressed as an infinite series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big] \qquad (1)$$

with the constants a_n and b_n given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

Furthermore, the Fourier series in Eq. (1) provides a $2L$-periodic extension of function f from the domain $[-L, L] \subset \mathbb{R}$ to $\mathbb{R}$.

The Fourier Theorem: Continuous case.

Sketch of the Proof:

- Define the partial sum functions

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big]$$

  with a_n and b_n given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

- Express f_N as a convolution of Sine, Cosine, functions and the original function f.
- Use the convolution properties to show that

$$\lim_{N\to\infty} f_N(x) = f(x), \qquad x \in [-L, L]. \qquad \square$$

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Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: In this case $L = 1$. The Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x) \big],$$

where the a_n, b_n are given in the Theorem. We start with a_0,

$$a_0 = \int_{-1}^{1} f(x)\, dx = \int_{-1}^{0} (1 + x)\, dx + \int_{0}^{1} (1 - x)\, dx.$$

$$a_0 = \Big(x + \frac{x^2}{2} \Big)\Big|_{-1}^{0} + \Big(x - \frac{x^2}{2} \Big)\Big|_{0}^{1} = \Big(1 - \frac{1}{2} \Big) + \Big(1 - \frac{1}{2} \Big)$$

We obtain: $a_0 = 1$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$. Similarly, the rest of the a_n are given by,

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx$$

$$a_n = \int_{-1}^{0} (1 + x) \cos(n\pi x)\, dx + \int_{0}^{1} (1 - x) \cos(n\pi x)\, dx.$$

Recall the integrals $\displaystyle\int \cos(n\pi x)\, dx = \frac{1}{n\pi} \sin(n\pi x)$, and

$$\int x \cos(n\pi x)\, dx = \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x).$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: It is not difficult to see that

$$a_n = \frac{1}{n\pi} \sin(n\pi x)\Big|_{-1}^{0} + \left[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x)\right]\Big|_{-1}^{0}$$
$$+ \frac{1}{n\pi} \sin(n\pi x)\Big|_{0}^{1} - \left[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x)\right]\Big|_{0}^{1}$$

$$a_n = \left[\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} \cos(-n\pi)\right] - \left[\frac{1}{n^2\pi^2} \cos(n\pi) - \frac{1}{n^2\pi^2}\right].$$

We then conclude that $a_n = \dfrac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$, and $a_n = \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]$.

Finally, we must find the coefficients b_n.

A similar calculation shows that $b_n = 0$.

Then, the Fourier series of f is given by

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]\,\cos(n\pi x).$$

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Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]\,\cos(n\pi x)$.

We can obtain a simpler expression for the Fourier coefficients a_n.

Recall the relations $\cos(n\pi) = (-1)^n$, then

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1 - (-1)^n\big]\,\cos(n\pi x).$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1 + (-1)^{n+1}\big]\,\cos(n\pi x).$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \dfrac{2}{n^2\pi^2} \left[1 + (-1)^{n+1}\right] \cos(n\pi x).$

If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2\pi^2}(1 - 1) \quad \Rightarrow \quad a_{2k} = 0.$$

If $n = 2k - 1$, so n is odd, so $n + 1 = 2k$ is even, then

$$a_{2k-1} = \frac{2}{(2k-1)^2\pi^2}(1 + 1) \quad \Rightarrow \quad a_{2k-1} = \frac{4}{(2k-1)^2\pi^2}.$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution:

Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \dfrac{2}{n^2\pi^2} \left[1 + (-1)^{n+1}\right] \cos(n\pi x)$, and

$$a_{2k} = 0, \qquad a_{2k-1} = \frac{4}{(2k-1)^2\pi^2}.$$

We conclude: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{k=1}^{\infty} \dfrac{4}{(2k-1)^2\pi^2} \cos((2k-1)\pi x).$ $\triangleleft$

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The Fourier Theorem: Piecewise continuous case.

Recall:

Definition

A function $f : [a, b] \to \mathbb{R}$ is called *piecewise continuous* iff holds,

(a) $[a, b]$ can be partitioned in a finite number of sub-intervals such that f is continuous on the interior of these sub-intervals.

(b) f has finite limits at the endpoints of all sub-intervals.

The Fourier Theorem: Piecewise continuous case.

Theorem (Fourier Series)

If $f : [-L, L] \subset \mathbb{R} \to \mathbb{R}$ is piecewise continuous, then the function

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big]$$

where a_n and b_n given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

satisfies that:

(a) $f_F(x) = f(x)$ *for all x where f is continuous;*

(b) $f_F(x_0) = \dfrac{1}{2}\Big[\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x) \Big]$ *for all x_0 where f is discontinuous.*

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Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: We start computing the Fourier coefficients b_n;

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad L = 1,$$

$$b_n = \int_{-1}^{0} (-1) \sin\big(n\pi x\big)\, dx + \int_{0}^{1} (1) \sin\big(n\pi x\big)\, dx,$$

$$b_n = \frac{(-1)}{n\pi}\Big[-\cos(n\pi x)\Big|_{-1}^{0}\Big] + \frac{1}{n\pi}\Big[-\cos(n\pi x)\Big|_{0}^{1}\Big],$$

$$b_n = \frac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \frac{1}{n\pi}\big[-\cos(n\pi) + 1\big].$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: $b_n = \frac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \frac{1}{n\pi}\big[-\cos(n\pi) + 1\big].$

$$b_n = \frac{1}{n\pi}\big[1 - \cos(-n\pi) - \cos(n\pi) + 1\big] = \frac{2}{n\pi}\big[1 - \cos(n\pi)\big],$$

We obtain: $b_n = \frac{2}{n\pi}\big[1 - (-1)^n\big].$

If $n = 2k$, then $b_{2k} = \frac{2}{2k\pi}\big[1 - (-1)^{2k}\big]$, hence $b_{2k} = 0$.

If $n = 2k - 1$, then $b_{2k-1} = \frac{2}{(2k-1)\pi}\big[1 - (-1)^{2k-1}\big]$,

hence $b_{2k} = \frac{4}{(2k-1)\pi}$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, and $b_{2k} = \dfrac{4}{(2k-1)\pi}$.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\cos\Big(\frac{n\pi x}{L}\Big)\,dx, \qquad L = 1,$$

$$a_n = \int_{-1}^{0} (-1)\,\cos\big(n\pi x\big)\,dx + \int_{0}^{1} (1)\,\cos\big(n\pi x\big)\,dx,$$

$$a_n = \frac{(-1)}{n\pi}\Big[\sin\big(n\pi x\big)\Big|_{-1}^{0}\Big] + \frac{1}{n\pi}\Big[\sin\big(n\pi x\big)\Big|_{0}^{1}\Big],$$

$$a_n = \frac{(-1)}{n\pi}\big[0 - \sin(-n\pi)\big] + \frac{1}{n\pi}\big[\sin(n\pi) - 0\big] \quad \Rightarrow \quad a_n = 0.$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, $b_{2k} = \dfrac{4}{(2k-1)\pi}$, and $a_n = 0$.

Therefore, we conclude that

$$f_F(x) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)}\,\sin\big((2k-1)\pi\,x\big).$$

◁