Review for Exam 3.

- 5 or 6 problems, 60 minutes.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Integration table an LT table provided.
- Exam covers:
  - Power Series with Regular-Singular points (3.3).
  - Chapter 4: Laplace Transform methods.
    - Definition of Laplace Transform (4.1).
    - Solving IVP using LT (4.2).
    - Solving IVP with discontinuous sources using LT, (4.3).
    - Solving IVP with generalized sources using LT (4.4).
    - Convolutions and LT (4.5).
  - Systems of linear Differential Equations (5.1).

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  - Systems of linear Differential Equations (5.1).

Regular-singular points (3.3).

Summary:

- Look for solutions $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)}$.
- Recall: Since $r \neq 0$, holds

  $$y' = \sum_{n=0}^{\infty} (n+r) a_n (x - x_0)^{(n+r-1)} \neq \sum_{n=1}^{\infty} (n+r) a_n (x - x_0)^{(n+r-1)},$$
- Find the indicial equation for r, the recurrence relation for a_n.
- Introduce the larger root r_+ of the indicial polynomial into the recurrence relation and solve for a_n.
  - (a) If $(r_+ - r_-)$ is not an integer, then each r_+ and r_- define linearly independent solutions.
  - (b) If $(r_+ - r_-)$ is an integer, then both r_+ and r_- define proportional solutions.

Regular-singular points (3.3).

Example

Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $y = \sum_{n=0}^{\infty} a_n x^{(n+r)}$, $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r-2)}$,

$$x^2 y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)}$$

We also need to compute

$$\left(x^2 + \frac{1}{4}\right) y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},$$

Regular-singular points (3.3).

Example

Consider the equation $x^2\, y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $\left(x^2 + \dfrac{1}{4}\right) y = \displaystyle\sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)}$.

Re-label $m = n + 2$ in the first term and then switch back to n,

$$\left(x^2 + \frac{1}{4}\right) y = \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},$$

The equation is

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.$$

Regular-singular points (3.3).

Example

Consider the equation $x^2\, y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution:

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.$$

$$\left[r(r-1) + \frac{1}{4}\right] a_0\, x^r + \left[(r+1)r + \frac{1}{4}\right] a_1\, x^{(r+1)} +$$

$$\sum_{n=2}^{\infty} \left[(n+r)(n+r-1) a_n + a_{(n-2)} + \frac{1}{4} a_n\right] x^{(n+r)} = 0.$$

Regular-singular points (3.3).

Example

Consider the equation $x^2\, y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $\left[r(r-1) + \dfrac{1}{4}\right] a_0 = 0, \quad \left[(r+1)r + \dfrac{1}{4}\right] a_1 = 0,$

$$\left[(n+r)(n+r-1) + \frac{1}{4}\right] a_n + a_{(n-2)} = 0.$$

The indicial equation $r^2 - r + \dfrac{1}{4} = 0$ implies $r_\pm = \dfrac{1}{2}$.

The indicial equation $r^2 + r + \dfrac{1}{4} = 0$ implies $r_\pm = -\dfrac{1}{2}$.

Choose $r = \dfrac{1}{2}$. That implies a_0 arbitrary and $a_1 = 0$.

Regular-singular points (3.3).

Example

Consider the equation $x^2\, y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $r = \dfrac{1}{2}, \quad a_1 = 0, \quad \left[(n+r)(n+r-1) + \dfrac{1}{4}\right] a_n = -a_{(n-2)}.$

$$\left[\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)+\frac{1}{4}\right] a_n = -a_{(n-2)} \;\Rightarrow\; \left[n^2 - \frac{1}{4} + \frac{1}{4}\right] a_n = -a_{(n-2)}$$

$$n^2 a_n = -a_{(n-2)} \;\Rightarrow\; a_n = -\frac{a_{(n-2)}}{n^2} \;\Rightarrow\; \begin{cases} a_2 = -\dfrac{a_0}{4}, \\ a_4 = -\dfrac{a_2}{16} = \dfrac{a_0}{64}. \end{cases}$$

Regular-singular points (3.3).

Example

Consider the equation $x^2\, y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $r = \dfrac{1}{2}$, $a_1 = 0$, $a_2 = -\dfrac{a_0}{4}$, and $a_4 = \dfrac{a_0}{64}$. Then,

$$y(x) = x^r\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\big).$$

Recall: $a_1 = 0$ and the recurrence relation imply $a_n = 0$ for n odd. Therefore,

$$y(x) = a_0 x^{1/2}\big(1 - \frac{1}{4}\,x^2 + \frac{1}{64}\,x^4 + \cdots\big).$$

$\triangleleft$

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Laplace transforms (Chptr. 4).

Summary:

- Main Properties:

$$\mathcal{L}\big[f^{(n)}(t)\big] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \quad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \quad (13)$$

$$\mathcal{L}[f(t)]\Big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]. \quad (14)$$

- Convolutions:

$$\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)].$$

- Partial fraction decompositions, completing the squares.

Chapter 4: Laplace Transform methods.

Example

Use Laplace Transform to find y solution of

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: Compute the LT of the equation,

$$\mathcal{L}[y''] - 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[\delta(t-2)] = e^{-2s}$$

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0), \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0).$$

$$(s^2 - 2s + 2)\,\mathcal{L}[y] - s\,y(0) - y'(0) + 2\,y(0) = e^{-2s}$$

$$(s^2 - 2s + 2)\,\mathcal{L}[y] - s - 1 = e^{-2s}$$

$$\mathcal{L}[y] = \frac{(s+1)}{(s^2 - 2s + 2)} + \frac{1}{(s^2 - 2s + 2)}\,e^{-2s}.$$

Chapter 4: Laplace Transform methods.

Example

Use Laplace Transform to find y solution of

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2 - 2s + 2)} + \dfrac{1}{(s^2 - 2s + 2)}\, e^{-2s}.$

$$s^2 - 2s + 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\big[2 \pm \sqrt{4-8}\big], \quad \text{complex roots.}$$

$$s^2 - 2s + 2 = (s^2 - 2s + 1) - 1 + 2 = (s-1)^2 + 1.$$

$$\mathcal{L}[y] = \frac{s+1}{(s-1)^2 + 1} + \frac{1}{(s-1)^2 + 1}\, e^{-2s}$$

$$\mathcal{L}[y] = \frac{(s-1+1)+1}{(s-1)^2 + 1} + \frac{1}{(s-1)^2 + 1}\, e^{-2s}$$

Chapter 4: Laplace Transform methods.

Example

Use Laplace Transform to find y solution of

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s-1)+2}{(s-1)^2 + 1} + \dfrac{1}{(s-1)^2 + 1}\, e^{-2s},$

$$\mathcal{L}[y] = \frac{(s-1)}{(s-1)^2 + 1} + 2\,\frac{1}{(s-1)^2 + 1} + e^{-2s}\,\frac{1}{(s-1)^2 + 1},$$

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2}, \qquad \mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2},$$

$$\mathcal{L}[y] = \mathcal{L}[\cos(t)]\big|_{(s-1)} + 2\,\mathcal{L}[\sin(t)]\big|_{(s-1)} + e^{-2s}\,\mathcal{L}[\sin(t)]\big|_{(s-1)}.$$

Chapter 4: Laplace Transform methods.

Example

Use Laplace Transform to find y solution of

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: Recall:

$$\mathcal{L}[y] = \mathcal{L}[\cos(t)]\big|_{(s-1)} + 2\,\mathcal{L}[\sin(t)]\big|_{(s-1)} + e^{-2s}\,\mathcal{L}[\sin(t)]\big|_{(s-1)}$$

and $\mathcal{L}[f(t)]\big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]$. Therefore,

$$\mathcal{L}[y] = \mathcal{L}[e^t \cos(t)] + 2\,\mathcal{L}[e^t \sin(t)] + e^{-2s}\,\mathcal{L}[e^t \sin(t)].$$

Also recall: $e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]$. Therefore,

$$\mathcal{L}[y] = \mathcal{L}[e^t \cos(t)] + 2\,\mathcal{L}[e^t \sin(t)] + \mathcal{L}[u_2(t)\,e^{(t-2)} \sin(t-2)].$$

$$y(t) = \big[\cos(t) + 2\sin(t)\big]\,e^t + u_2(t)\,\sin(t-2)\,e^{(t-2)}. \qquad \triangleleft$$

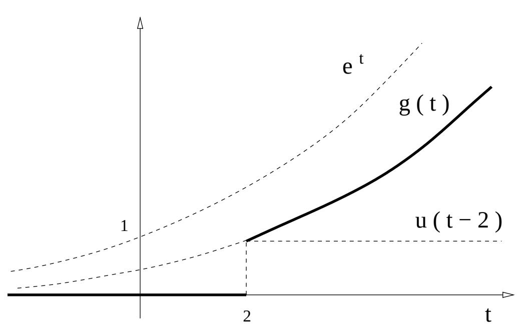
Chapter 4: Laplace Transform methods.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution:



Express g using step functions,

$$g(t) = u_2(t)\,e^{(t-2)}.$$

$$\mathcal{L}[u_c(t)\,f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)].$$

Therefore,

$$\mathcal{L}[g(t)] = e^{-2s}\mathcal{L}[e^t].$$

We obtain: $\mathcal{L}[g(t)] = \dfrac{e^{-2s}}{(s-1)}$.

Chapter 4: Laplace Transform methods.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution: Recall: $\mathcal{L}[g(t)] = \dfrac{e^{-2s}}{(s-1)}$.

$$\mathcal{L}[y''] + 3\,\mathcal{L}[y] = \mathcal{L}[g(t)] = \frac{e^{-2s}}{(s-1)}.$$

$$(s^2+3)\,\mathcal{L}[y] = \frac{e^{-2s}}{(s-1)} \quad \Rightarrow \quad \mathcal{L}[y] = e^{-2s}\,\frac{1}{(s-1)(s^2+3)}.$$

$$H(s) = \frac{1}{(s-1)(s^2+3)} = \frac{a}{(s-1)} + \frac{(bs+c)}{(s^2+3)}$$

$$1 = a(s^2+3) + (bs+c)(s-1)$$

Chapter 4: Laplace Transform methods.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution: Recall: $1 = a(s^2+3) + (bs+c)(s-1)$.

$$1 = as^2 + 3a + bs^2 + cs - bs - c$$

$$1 = (a+b)\,s^2 + (c-b)\,s + (3a-c)$$

$$a+b=0, \quad c-b=0, \quad 3a-c=1.$$

$$a=-b, \quad c=b, \quad -3b-b=1 \quad \Rightarrow \quad b=-\frac{1}{4},\ a=\frac{1}{4},\ c=-\frac{1}{4}.$$

$$H(s) = \frac{1}{4}\Big[\frac{1}{s-1} - \frac{s+1}{s^2+3}\Big].$$

Chapter 4: Laplace Transform methods.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution: Recall: $H(s) = \frac{1}{4}\Big[\frac{1}{s-1} - \frac{s+1}{s^2+3}\Big]$, $\mathcal{L}[y] = e^{-2s}\,H(s)$.

$$H(s) = \frac{1}{4}\Big[\frac{1}{s-1} - \frac{s}{s^2+3} - \frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{s^2+3}\Big],$$

$$H(s) = \frac{1}{4}\Big[\mathcal{L}[e^t] - \mathcal{L}\big[\cos\big(\sqrt{3}\,t\big)\big] - \frac{1}{\sqrt{3}}\,\mathcal{L}\big[\sin\big(\sqrt{3}\,t\big)\big]\Big].$$

$$H(s) = \mathcal{L}\Big[\frac{1}{4}\Big(e^t - \cos\big(\sqrt{3}\,t\big) - \frac{1}{\sqrt{3}}\,\sin\big(\sqrt{3}\,t\big)\Big)\Big].$$

Chapter 4: Laplace Transform methods.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution: Recall: $H(s) = \mathcal{L}\Big[\frac{1}{4}\Big(e^t - \cos\big(\sqrt{3}\,t\big) - \frac{1}{\sqrt{3}}\,\sin\big(\sqrt{3}\,t\big)\Big)\Big]$.

$$h(t) = \frac{1}{4}\Big(e^t - \cos\big(\sqrt{3}\,t\big) - \frac{1}{\sqrt{3}}\,\sin\big(\sqrt{3}\,t\big)\Big), \quad H(s) = \mathcal{L}[h(t)].$$

$$\mathcal{L}[y] = e^{-2s}\,H(s) = e^{-2s}\,\mathcal{L}[h(t)] = \mathcal{L}[u_2(t)\,h(t-2)].$$

We conclude: $y(t) = u_2(t)\,h(t-2)$. Equivalently,

$$y(t) = \frac{u_2(t)}{4}\,\Big[e^{(t-2)} - \cos\big(\sqrt{3}\,(t-2)\big) - \frac{1}{\sqrt{3}}\,\sin\big(\sqrt{3}\,(t-2)\big)\Big].$$

◁

Chapter 4: Laplace Transform methods.

Example

Use convolutions to find f satisfying $\mathcal{L}[f(t)] = \dfrac{e^{-2s}}{(s-1)(s^2+3)}$.

Solution: One way to solve this is with the splitting

$$\mathcal{L}[f(t)] = e^{-2s}\,\frac{1}{(s^2+3)}\,\frac{1}{(s-1)} = e^{-2s}\,\frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{(s^2+3)}\,\frac{1}{(s-1)},$$

$$\mathcal{L}[f(t)] = e^{-2s}\,\frac{1}{\sqrt{3}}\,\mathcal{L}[\sin(\sqrt{3}\,t)]\,\mathcal{L}[e^t]$$

$$\mathcal{L}[f(t)] = \frac{1}{\sqrt{3}}\mathcal{L}[u_2(t)\,\sin(\sqrt{3}\,(t-2))]\,\mathcal{L}[e^t].$$

$$f(t) = \frac{1}{\sqrt{3}}\int_0^t u_2(\tau)\,\sin(\sqrt{3}\,(\tau-2))\,e^{(t-\tau)}\,d\tau. \qquad \triangleleft$$

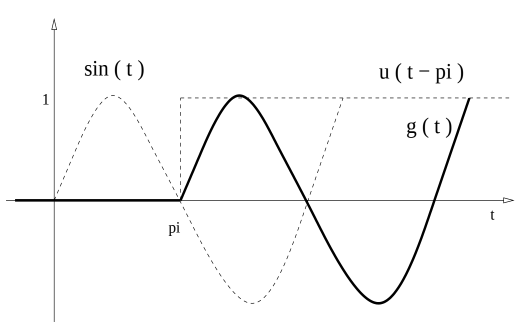
Chapter 4: Laplace Transform methods.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t-\pi), & t \geqslant \pi. \end{cases}$$

Solution:



Express g using step functions,

$$g(t) = u_\pi(t)\,\sin(t-\pi).$$

$$\mathcal{L}[u_c(t)\,f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)].$$

Therefore,

$$\mathcal{L}[g(t)] = e^{-\pi s}\mathcal{L}[\sin(t)].$$

We obtain: $\mathcal{L}[g(t)] = \dfrac{e^{-\pi s}}{s^2+1}$.

Chapter 4: Laplace Transform methods.

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$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geqslant \pi. \end{cases}$$

Solution: $\mathcal{L}[g(t)] = \dfrac{e^{-\pi s}}{s^2 + 1}$.

$$\mathcal{L}[y''] - 6\,\mathcal{L}[y] = \mathcal{L}[g(t)] = \frac{e^{-\pi s}}{s^2 + 1}.$$

$$(s^2 - 6)\,\mathcal{L}[y] = \frac{e^{-\pi s}}{s^2 + 1} \quad \Rightarrow \quad \mathcal{L}[y] = e^{-\pi s}\,\frac{1}{(s^2 + 1)(s^2 - 6)}.$$

$$H(s) = \frac{1}{(s^2 + 1)(s^2 - 6)} = \frac{1}{(s^2 + 1)(s + \sqrt{6})(s - \sqrt{6})}$$

$$H(s) = \frac{a}{(s + \sqrt{6})} + \frac{b}{(s - \sqrt{6})} + \frac{(cs + d)}{(s^2 + 1)}.$$

Chapter 4: Laplace Transform methods.

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Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geqslant \pi. \end{cases}$$

Solution: $H(s) = \dfrac{a}{(s + \sqrt{6})} + \dfrac{b}{(s - \sqrt{6})} + \dfrac{(cs + d)}{(s^2 + 1)}$.

$$\frac{1}{(s^2 + 1)(s + \sqrt{6})(s - \sqrt{6})} = \frac{a}{(s + \sqrt{6})} + \frac{b}{(s - \sqrt{6})} + \frac{(cs + d)}{(s^2 + 1)}$$

$$1 = a(s - \sqrt{6})(s^2 + 1) + b(s + \sqrt{6})(s^2 + 1) + (cs + d)(s^2 - 6).$$

The solution is: $a = -\dfrac{1}{14\sqrt{6}}, \;\; b = \dfrac{1}{14\sqrt{6}}, \;\; c = 0, \;\; d = -\dfrac{1}{7}.$

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geqslant \pi. \end{cases}$$

Solution: $H(s) = \dfrac{1}{14\sqrt{6}} \Big[-\dfrac{1}{(s+\sqrt{6})} + \dfrac{1}{(s-\sqrt{6})} - \dfrac{2\sqrt{6}}{(s^2+1)} \Big].$

$$H(s) = \frac{1}{14\sqrt{6}} \Big[-\mathcal{L}\big[e^{-\sqrt{6}\,t}\big] + \mathcal{L}\big[e^{\sqrt{6}\,t}\big] - 2\sqrt{6}\,\mathcal{L}[\sin(t)] \Big]$$

$$H(s) = \mathcal{L}\Big[\frac{1}{14\sqrt{6}} \Big(-e^{-\sqrt{6}\,t} + e^{\sqrt{6}\,t} - 2\sqrt{6}\,\sin(t) \Big) \Big].$$

$$h(t) = \frac{1}{14\sqrt{6}} \Big[-e^{-\sqrt{6}\,t} + e^{\sqrt{6}\,t} - 2\sqrt{6}\,\sin(t) \Big] \;\Rightarrow\; H(s) = \mathcal{L}[h(t)].$$

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geqslant \pi. \end{cases}$$

Solution: Recall: $\mathcal{L}[y] = e^{-\pi s}\,H(s)$, where $H(s) = \mathcal{L}[h(t)]$, and

$$h(t) = \frac{1}{14\sqrt{6}} \Big[-e^{-\sqrt{6}\,t} + e^{\sqrt{6}\,t} - 2\sqrt{6}\,\sin(t) \Big].$$

$$\mathcal{L}[y] = e^{-\pi s}\,\mathcal{L}[h(t)] = \mathcal{L}[u_\pi(t)\,h(t-\pi)] \;\Rightarrow\; y(t) = u_\pi(t)\,h(t-\pi).$$

Equivalently:

$$y(t) = \frac{u_\pi(t)}{14\sqrt{6}} \Big[-e^{-\sqrt{6}\,(t-\pi)} + e^{\sqrt{6}\,(t-\pi)} - 2\sqrt{6}\,\sin(t-\pi) \Big]. \quad \triangleleft$$

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  - **Systems of linear Differential Equations (5.1).**

Second order equations and first order systems.

Theorem (Reduction to first order)

Every solution y to the second order linear equation

$$y'' + p(t)\,y' + q(t)\,y = g(t), \tag{1}$$

defines a solution $x_1 = y$ and $x_2 = y'$ of the 2×2 first order linear differential system

$$x_1' = x_2, \tag{2}$$

$$x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t). \tag{3}$$

Conversely, every solution x_1, x_2 of the 2×2 first order linear system in Eqs. (2)-(3) defines a solution $y = x_1$ of the second order differential equation in (1).

Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Compute x_1 from the second equation: $x_1 = x_2' + x_2$.
Introduce this expression into the first equation,

$$(x_2' + x_2)' = -(x_2' + x_2) + 3x_2,$$

$$x_2'' + x_2' = -x_2' - x_2 + 3x_2,$$

$$x_2'' + 2x_2' - 2x_2 = 0.$$

Second order equations and first order systems.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Recall: $x_2'' + 2x_2' - 2x_2 = 0$.

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4+8}\right] \quad \Rightarrow \quad r_\pm = -1 \pm \sqrt{3}.$$

Therefore, $x_2 = c_1\, e^{r_+ t} + c_2\, e^{r_- t}$. Since $x_1 = x_2' + x_2$,

$$x_1 = \left(c_1 r_+\, e^{r_+ t} + c_2 r_-\, e^{r_- t}\right) + \left(c_1\, e^{r_+ t} + c_2\, e^{r_- t}\right),$$

We conclude: $x_1 = c_1(1 + r_+)\, e^{r_+ t} + c_2(1 + r_-)\, e^{r_- t}$. $\triangleleft$