Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: Eigenvalue-eigenfunction problem.

Two-point Boundary Value Problem.

Definition
A *two-point BVP* is the following: Given functions p, q, g, and constants
$$x_1 < x_2, \quad y_1, y_2, \quad b_1, b_2, \quad \tilde{b}_1, \tilde{b}_2,$$
find a function y solution of the differential equation
$$y'' + p(x)\, y' + q(x)\, y = g(x),$$
together with the extra, *boundary conditions*,
$$b_1\, y(x_1) + b_2\, y'(x_1) = y_1,$$
$$\tilde{b}_1\, y(x_2) + \tilde{b}_2\, y'(x_2) = y_2.$$

Remarks:

- Both y and y' might appear in the boundary condition, evaluated at the same point.
- In this notes we only study the case of constant coefficients,
$$y'' + a_1\, y' + a_0\, y = g(x).$$

Two-point Boundary Value Problem.

Example

Examples of BVP. Assume $x_1 \neq x_2$.

(1) Find y solution of

$$y'' + a_1\, y' + a_0\, y = g(x), \quad y(x_1) = y_1, \quad y(x_2) = y_2.$$

(2) Find y solution of

$$y'' + a_1\, y' + a_0\, y = g(x), \quad y'(x_1) = y_1, \quad y'(x_2) = y_2.$$

(3) Find y solution of

$$y'' + a_1\, y' + a_0\, y = g(x), \quad y(x_1) = y_1, \quad y'(x_2) = y_2.$$

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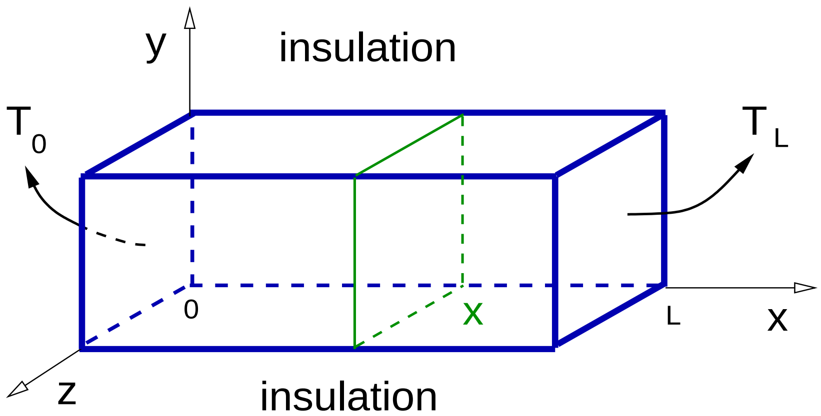
Example from physics.

Problem: The equilibrium (time independent) temperature of a bar of length L with insulated horizontal sides and the bar vertical extremes kept at fixed temperatures T_0, T_L is the solution of the BVP:

$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$



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Comparison: IVP vs BVP.

Review: IVP:
Find the function values $y(t)$ solutions of the differential equation

$$y'' + a_1\, y' + a_0\, y = g(t),$$

together with the initial conditions

$$y(t_0) = y_1, \quad y'(t_0) = y_2.$$

Remark: In physics:

- $y(t)$: Position at time t.
- Initial conditions: Position and velocity at the initial time t_0.

Comparison: IVP vs BVP.

Review: BVP:
Find the function values $y(x)$ solutions of the differential equation

$$y'' + a_1\, y' + a_0\, y = g(x),$$

together with the initial conditions

$$y(x_1) = y_1, \quad y(x_2) = y_2.$$

Remark: In physics:

- $y(x)$: A physical quantity (temperature) at a position x.
- Boundary conditions: Conditions at the boundary of the object under study, where $x_1 \neq x_2$.

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Existence, uniqueness of solutions to BVP.

Review: The initial value problem.

Theorem (IVP)

Consider the homogeneous initial value problem:

$$y'' + a_1\, y' + a_0\, y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_1,$$

and let $r_\pm$ *be the roots of the characteristic polynomial*

$$p(r) = r^2 + a_1\, r + a_0.$$

If $r_+ \neq r_-$*, real or complex, then for every choice of* y_0*,* y_1*, there exists a unique solution* y *to the initial value problem above.*

Summary: The IVP above always has a unique solution, no matter what y_0 and y_1 we choose.

Existence, uniqueness of solutions to BVP.

Theorem (BVP)

Consider the homogeneous boundary value problem:

$$y'' + a_1\, y' + a_0\, y = 0, \qquad y(0) = y_0, \quad y(L) = y_1,$$

and let $r_\pm$ be the roots of the characteristic polynomial

$$p(r) = r^2 + a_1\, r + a_0.$$

(A) *If $r_+ \neq r_-$, real, then for every choice of $L \neq 0$ and y_0, y_1, there exists a unique solution y to the BVP above.*

(B) *If $r_\pm = \alpha \pm i\beta$, with $\beta \neq 0$, and $\alpha, \beta \in \mathbb{R}$, then the solutions to the BVP above belong to one of these possibilities:*

  (1) *There exists a unique solution.*
  (2) *There exists no solution.*
  (3) *There exist infinitely many solutions.*

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1\, e^{r_-\, t} + c_2\, e^{r_+\, t}, \qquad c_1, c_2 \in \mathbb{R}.$$

The initial conditions determine c_1 and c_2 as follows:

$$y_0 = y(t_0) = c_1\, e^{r_-\, t_0} + c_2\, e^{r_+\, t_0}$$

$$y_1 = y'(t_0) = c_1 r_-\, e^{r_-\, t_0} + c_2 r_+\, e^{r_+\, t_0}$$

Using matrix notation,

$$\begin{bmatrix} e^{r_-\, t_0} & e^{r_+\, t_0} \\ r_-\, e^{r_-\, t_0} & r_+\, e^{r_+\, t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the $\det(Z) \neq 0$, where

$$Z = \begin{bmatrix} e^{r_-\, t_0} & e^{r_+\, t_0} \\ r_-\, e^{r_-\, t_0} & r_+\, e^{r_+\, t_0} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

Existence, uniqueness of solutions to BVP.

Proof of IVP:

Recall: $Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = (r_+ - r_-)\, e^{(r_+ + r_-)\, t_0} \neq 0 \quad \Leftrightarrow \quad r_+ \neq r_-.$$

Since $r_+ \neq r_-$, the matrix Z is invertible and so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Z^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

We conclude that for every choice of y_0 and y_1, there exist a unique value of c_1 and c_2, so the IVP above has a unique solution. $\square$

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1\, e^{r_- x} + c_2\, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine c_1 and c_2 as follows:

$$y_0 = y(0) = c_1 + c_2.$$

$$y_1 = y(L) = c_1\, e^{r_- L} + c_2\, e^{r_+ L}$$

Using matrix notation,

$$\begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the $\det(Z) \neq 0$, where

$$Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$.

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \quad \Leftrightarrow \quad e^{r_+ L} \neq e^{r_- L}.$$

(A) If $r_+ \neq r_-$ and real-valued, then $\det(Z) \neq 0$.

We conclude: For every choice of y_0 and y_1, there exist a unique value of c_1 and c_2, so the BVP in (A) above has a unique solution.

(B) If $r_\pm = \alpha \pm i\beta$, with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$, then

$$\det(Z) = e^{\alpha L}\big(e^{i\beta L} - e^{-i\beta L}\big) \;\Rightarrow\; \det(Z) = 2i\, e^{\alpha L} \sin(\beta L).$$

Since $\det(Z) = 0$ iff $\beta L = n\pi$, with n integer,

(1) If $\beta L \neq n\pi$, then BVP has a unique solution.
(2) If $\beta L = n\pi$ then BVP either has no solutions or it has infinitely many solutions. □

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 1, \quad c_2 \text{ free.}$$

We conclude: $y(x) = \cos(x) + c_2 \sin(x)$, with $c_2 \in \mathbb{R}$.

The BVP has infinitely many solutions. ◁

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 0 = y(\pi) = -c_1$$

The BVP has no solution. $\triangleleft$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2 \quad \Rightarrow \quad c_1 = c_2 = 1.$$

We conclude: $y(x) = \cos(x) + \sin(x)$.

The BVP has a unique solution. $\triangleleft$

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Particular case of BVP: Eigenvalue-eigenfunction problem.

Problem:
Find a number λ and a non-zero function y solutions to the boundary value problem

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remark: This problem is similar to the eigenvalue-eigenvector problem in Linear Algebra: Given an $n \times n$ matrix A, find λ and a non-zero n-vector $\mathbf{v}$ solutions of

$$A\mathbf{v} - \lambda\,\mathbf{v} = \mathbf{0}.$$

Differences:

- $A \quad \longrightarrow \quad \left\{ \begin{array}{l} \text{computing a second derivative and} \\ \text{applying the boundary conditions.} \end{array} \right\}$
- $\mathbf{v} \quad \longrightarrow \quad \{\text{a function } y\}.$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remarks: We will show that:

(1) If $\lambda \leqslant 0$, then the BVP has no solution.

(2) If $\lambda > 0$, then there exist infinitely many eigenvalues λ_n and eigenfunctions y_n, with n any positive integer, given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right),$$

(3) Analogous results can be proven for the same equation but with different types of boundary conditions. For example, for $y(0) = 0,\ y'(L) = 0$; or for $y'(0) = 0,\ y'(L) = 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda = 0$. The equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x.$$

The boundary conditions imply

$$0 = y(0) = c_1, \quad 0 = c_1 + c_2 L \quad \Rightarrow \quad c_1 = c_2 = 0.$$

Since $y = 0$, there are NO non-zero solutions for $\lambda = 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda < 0$. Introduce the notation $\lambda = -\mu^2$. The characteristic equation is

$$p(r) = r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu.$$

The general solution is

$$y(x) = c_1\, e^{\mu x} + c_2\, e^{-\mu x}.$$

The boundary condition are

$$0 = y(0) = c_1 + c_2,$$

$$0 = y(L) = c_1\, e^{\mu L} + c_2\, e^{-\mu L}.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $y(x) = c_1\, e^{\mu x} + c_2\, e^{\mu x}$ and

$$c_1 + c_2 = 0, \qquad c_1\, e^{\mu L} + c_2\, e^{-\mu L} = 0.$$

We need to solve the linear system

$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;\Leftrightarrow\; Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix}$$

Since $\det(Z) = e^{-\mu L} - e^{\mu L} \neq 0$ for $L \neq 0$, matrix Z is invertible, so the linear system above has a unique solution $c_1 = 0$ and $c_2 = 0$.

Since $y = 0$, there are NO non-zero solutions for $\lambda < 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

The boundary condition are

$$0 = y(0) = c_1, \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(L) = c_2 \sin(\mu L), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu L) = 0.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

The non-zero solution condition is the reason for $c_2 \neq 0$. Hence

$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L}.$$

Recalling that $\lambda_n = \mu_n^2$, and choosing $c_2 = 1$, we conclude

$$\lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \qquad y_n(x) = \sin\Big(\frac{n\pi x}{L}\Big). \qquad \triangleleft$$