## Review of Linear Algebra (Sect. 5.5)

- Eigenvalues, eigenvectors of a matrix.
- Computing eigenvalues and eigenvectors.
- Diagonalizable matrices.
- The case of Hermitian matrices.

## Eigenvalues, eigenvectors of a matrix

### Definition

A number $\lambda$ and a non-zero $n$-vector $\mathbf{v}$ are respectively called an *eigenvalue* and *eigenvector* of an $n \times n$ matrix $A$ iff the following equation holds,

$$A\mathbf{v} = \lambda \mathbf{v}\,.$$

### Example

Verify that the pair $\lambda_1 = 4$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = -2$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvalue and eigenvector pairs of matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: $A\mathbf{v}_1 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \mathbf{v}_1$.

$$A\mathbf{v}_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_2 \mathbf{v}_2. \qquad \triangleleft$$

## Eigenvalues, eigenvectors of a matrix

**Remarks:**

- If we interpret an $n \times n$ matrix $A$ as a function $A : \mathbb{R}^n \to \mathbb{R}^n$, then the eigenvector $\mathbf{v}$ determines a particular *direction* on $\mathbb{R}^n$ where the action of $A$ is *simple*: $A\mathbf{v}$ is proportional to $\mathbf{v}$.
- Matrices usually change the direction of the vector, like

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

- This is not the case for eigenvectors, like

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

## Eigenvalues, eigenvectors of a matrix

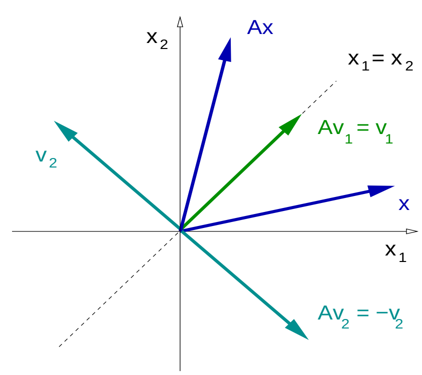
**Example**

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

**Solution:**

The function $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a reflection along $x_1 = x_2$ axis.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$




The line $x_1 = x_2$ is invariant under $A$. Hence,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = 1.$$

An eigenvalue eigenvector pair is: $\lambda_1 = 1$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

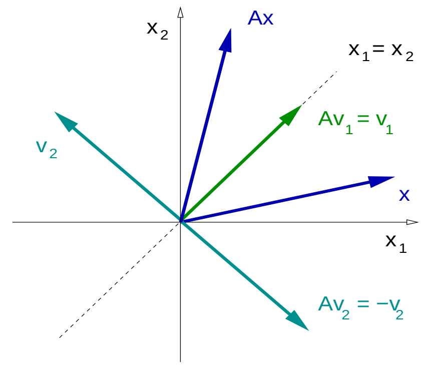
## Eigenvalues, eigenvectors of a matrix

**Example**

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution: Eigenvalue eigenvector pair:

$$\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$




A second eigenvector eigenvalue pair is:

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \lambda_2 = -1.$$

A second eigenvalue eigenvector pair: $\lambda_2 = -1$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $\triangleleft$

## Eigenvalues, eigenvectors of a matrix

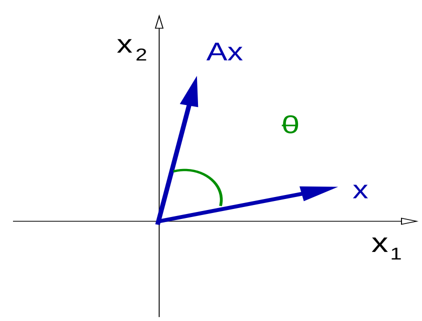
Remark: Not every $n \times n$ matrix has real eigenvalues.

**Example**

Fix $\theta \in (0, \pi)$ and define $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

Show that $A$ has no real eigenvalues.

Solution: Matrix $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation by $\theta$ counterclockwise.
There is no direction left invariant by the function $A$.




We conclude: Matrix $A$ has no eigenvalues eigenvector pairs. $\triangleleft$

Remark:
Matrix $A$ has complex-values eigenvalues and eigenvectors.

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- **Computing eigenvalues and eigenvectors.**
- Diagonalizable matrices.
- The case of Hermitian matrices.

## Computing eigenvalues and eigenvectors.

Problem:
Given an $n \times n$ matrix $A$, find, if possible, $\lambda$ and $\mathbf{v} \neq \mathbf{0}$ solution of

$$A\mathbf{v} = \lambda\,\mathbf{v}.$$

Remark:
This is more complicated than solving a linear system $A\mathbf{v} = \mathbf{b}$, since in our case we do not know the source vector $\mathbf{b} = \lambda\mathbf{v}$.

Solution:

(a) First solve for $\lambda$.

(b) Having $\lambda$, then solve for $\mathbf{v}$.

## Computing eigenvalues and eigenvectors.

### Theorem (Eigenvalues-eigenvectors)

(a) *The number $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ iff*

$$\det(A - \lambda I) = 0.$$

(b) *Given an eigenvalue $\lambda$ of matrix $A$, the corresponding eigenvectors $\mathbf{v}$ are the non-zero solutions to the homogeneous linear system*

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Notation:
$p(\lambda) = \det(A - \lambda I)$ is called the *characteristic polynomial*.
If $A$ is $n \times n$, then $p$ is degree $n$.

Remark: An eigenvalue is a root of the characteristic polynomial.

## Computing eigenvalues and eigenvectors.

Proof:
Find $\lambda$ such that for a non-zero vector $\mathbf{v}$ holds,

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Leftrightarrow \quad (A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Recall, $\mathbf{v} \neq \mathbf{0}$.

This last condition implies that matrix $(A - \lambda I)$ is not invertible.

(Proof: If $(A - \lambda I)$ invertible, then $(A - \lambda I)^{-1}(A - \lambda I)\mathbf{v} = \mathbf{0}$, that is, $\mathbf{v} = \mathbf{0}$.)

Since $(A - \lambda I)$ is not invertible, then $\det(A - \lambda I) = 0$.

Once $\lambda$ is known, the original eigenvalue-eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $(A - \lambda I)\mathbf{v} = \mathbf{0}$. $\square$

## Computing eigenvalues and eigenvectors.

Example

Find the eigenvalues $\lambda$ and eigenvectors $\mathbf{v}$ of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution:

The eigenvalues are the roots of the characteristic polynomial.

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{bmatrix}$$

The characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9$$

The roots are $\lambda_+ = 4$ and $\lambda_- = -2$.

Compute the eigenvector for $\lambda_+ = 4$. Solve $(A - 4I)\mathbf{v}_+ = \mathbf{0}$.

$$A - 4I = \begin{bmatrix} 1-4 & 3 \\ 3 & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}.$$

## Computing eigenvalues and eigenvectors.

Example

Find the eigenvalues $\lambda$ and eigenvectors $\mathbf{v}$ of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = 4$, $\lambda_- = -2$, $A - 4I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$.

We solve $(A - 4I)\mathbf{v}_+ = \mathbf{0}$, using Gauss elimination,

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^+ = v_2^+, \\ v_2^+ \quad \text{free.} \end{cases}$$

Al solutions to the equation above are then given by

$$\mathbf{v}_+ = \begin{bmatrix} v_2^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2^+ \quad \Rightarrow \quad \mathbf{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

The first eigenvalue eigenvector pair is $\lambda_+ = 4$, $\mathbf{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

## Computing eigenvalues and eigenvectors.

Example

Find the eigenvalues $\lambda$ and eigenvectors $\mathbf{v}$ of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = 4$, $\mathbf{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda_- = -2$.

Solve $(A + 2I)\mathbf{v}_- = \mathbf{0}$, using Gauss operations on $A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^- = -v_2^-, \\ v_2^- \quad \text{free.} \end{cases}$$

Al solutions to the equation above are then given by

$$\mathbf{v}_- = \begin{bmatrix} -v_2^- \\ v_2^- \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2^- \quad \Rightarrow \quad \mathbf{v}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

The second eigenvalue eigenvector pair: $\lambda_- = -2$, $\mathbf{v}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $\triangleleft$

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## Diagonalizable matrices.

### Definition

An $n \times n$ matrix $D$ is called *diagonal* iff $D = \begin{bmatrix} d_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_{nn} \end{bmatrix}$.

### Definition

An $n \times n$ matrix $A$ is called *diagonalizable* iff there exists an invertible matrix $P$ and a diagonal matrix $D$ such that

$$A = PDP^{-1}.$$

Remark:

- Systems of linear *differential* equations are simple to solve in the case that the coefficient matrix $A$ is diagonalizable.
- In such case, it is simple to *decouple* the differential equations.
- One solves the decoupled equations, and then transforms back to the original unknowns.

## Diagonalizable matrices.

### Theorem (Diagonalizability and eigenvectors)

*An $n \times n$ matrix $A$ is diagonalizable iff matrix $A$ has a linearly independent set of $n$ eigenvectors. Furthermore,*

$$A = PDP^{-1}, \quad P = [\mathbf{v}_1, \cdots, \mathbf{v}_n], \quad D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix},$$

*where $\lambda_i, \mathbf{v}_i$, for $i = 1, \cdots, n$, are eigenvalue-eigenvector pairs of $A$.*

Remark: It is not simple to know whether an $n \times n$ matrix $A$ has a linearly independent set of $n$ eigenvectors. One simple case is given in the following result.

### Theorem ($n$ different eigenvalues)

*If an $n \times n$ matrix $A$ has $n$ different eigenvalues, then $A$ is diagonalizable.*

## Diagonalizable matrices.

Example

Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Solution: We known that the eigenvalue eigenvector pairs are

$$\lambda_1 = 4, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -2, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Introduce $P$ and $D$ as follows,

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

Then

$$PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

## Diagonalizable matrices.

Example

Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Solution: Recall:

$$PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$PDP^{-1} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

We conclude,

$$PDP^{-1} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = A,$$

that is, $A$ is diagonalizable. $\triangleleft$

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## The case of Hermitian matrices.

Definition
An $n \times n$ matrix $A$ is called Hermitian iff $A = A^*$.
An $n \times n$ matrix $A$ is called symmetric iff $A = A^T$.

Theorem
*Every Hermitian matrix is diagonalizable.*

Remark: A real-valued Hermitian matrix $A$ is symmetric, since

$$A = A^* = \overline{A}^T = A^T \quad \Rightarrow \quad A = A^T$$

Example

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 7 \\ 3 & 7 & 11 \end{bmatrix}$ is symmetric, $B = \begin{bmatrix} 1 & -i & 1 \\ i & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ is Hermitian.