

Generalized sources (Sect. 4.4).

- ▶ The Dirac delta generalized function.
- ▶ Properties of Dirac's delta.
- ▶ Relation between deltas and steps.
- ▶ Dirac's delta in Physics.
- ▶ The Laplace Transform of Dirac's delta.
- ▶ Differential equations with Dirac's delta sources.

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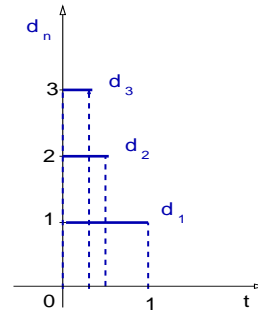
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The Dirac delta generalized function.

Definition

Consider the sequence of functions for $n \geq 1$,

$$\delta_n(t) = \begin{cases} 0, & t < 0 \\ n, & 0 \leq t \leq \frac{1}{n} \\ 0, & t > \frac{1}{n}. \end{cases}$$



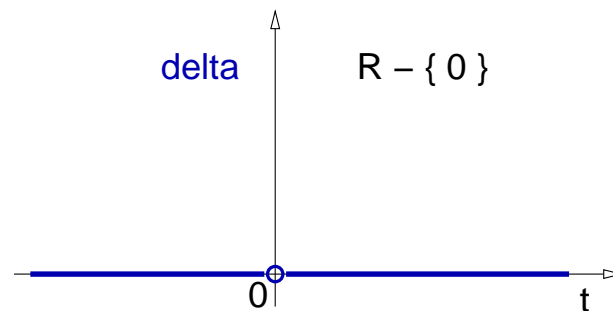
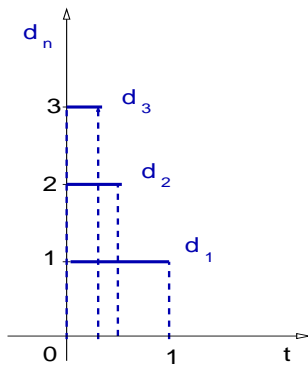
The Dirac delta generalized function is given by

$$\lim_{n \rightarrow \infty} \delta_n(t) = \delta(t), \quad t \in \mathbb{R}.$$

Remarks:

- (a) There exist infinitely many sequences δ_n that define the same generalized function δ .
- (b) For example, compare with the sequences δ_n in the literature.

The Dirac delta generalized function.



Remarks:

- (a) The Dirac δ is a function on the domain $\mathbb{R} - \{0\}$, and $\delta(t) = 0$ for $t \in \mathbb{R} - \{0\}$.
- (b) δ at $t = 0$ is not defined, since $\delta(0) = \lim_{n \rightarrow \infty} n = +\infty$.
- (c) δ is not a function on \mathbb{R} .

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Properties of Dirac's delta.

Remark: The Dirac δ is not a function on \mathbb{R} .

We define operations on Dirac's δ as limits $n \rightarrow \infty$ of the operation on the sequence elements δ_n .

Definition

$$\delta(t - c) = \lim_{n \rightarrow \infty} \delta_n(t - c),$$

$$a\delta(t) + b\delta(t) = \lim_{n \rightarrow \infty} [a\delta_n(t) + b\delta_n(t)],$$

$$f(t)\delta(t) = \lim_{n \rightarrow \infty} [f(t)\delta_n(t)],$$

$$\int_a^b \delta(t) dt = \lim_{n \rightarrow \infty} \int_a^b \delta_n(t) dt,$$

$$\mathcal{L}[\delta] = \lim_{n \rightarrow \infty} \mathcal{L}[\delta_n].$$

Properties of Dirac's delta.

Theorem

$$\int_{-a}^a \delta(t) dt = 1, \quad a > 0.$$

Proof:

$$\int_{-a}^a \delta(t) dt = \lim_{n \rightarrow \infty} \int_{-a}^a \delta_n(t) dt = \lim_{n \rightarrow \infty} \int_0^{1/n} n dt$$

$$\int_{-a}^a \delta(t) dt = \lim_{n \rightarrow \infty} \left[n \left(t \Big|_0^{1/n} \right) \right] = \lim_{n \rightarrow \infty} \left[n \frac{1}{n} \right].$$

We conclude: $\int_{-a}^a \delta(t) dt = 1.$

□

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) dt = f(t_0).$$

Proof: Introduce the change of variable $\tau = t - t_0$,

$$I = \int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) dt = \int_{-a}^a \delta(\tau) f(\tau + t_0) d\tau,$$

$$I = \lim_{n \rightarrow \infty} \int_{-a}^a \delta_n(\tau) f(\tau + t_0) d\tau = \lim_{n \rightarrow \infty} \int_0^{1/n} n f(\tau + t_0) d\tau$$

Therefore, $I = \lim_{n \rightarrow \infty} n \int_0^{1/n} F'(\tau + t_0) d\tau$, where we introduced the primitive $F(t) = \int f(t) dt$, that is, $f(t) = F'(t)$.

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) dt = f(t_0).$$

Proof: So, $I = \lim_{n \rightarrow \infty} n \int_0^{1/n} F'(\tau + t_0) d\tau$, with $f(t) = F'(t)$.

$$I = \lim_{n \rightarrow \infty} n \left[F(\tau + t_0) \Big|_0^{1/n} \right] = \lim_{n \rightarrow \infty} n \left[F\left(t_0 + \frac{1}{n}\right) - F(t_0) \right].$$

$$I = \lim_{n \rightarrow \infty} \frac{\left[F\left(t_0 + \frac{1}{n}\right) - F(t_0) \right]}{\frac{1}{n}} = F'(t_0) = f(t_0).$$

We conclude: $\int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) dt = f(t_0)$. □

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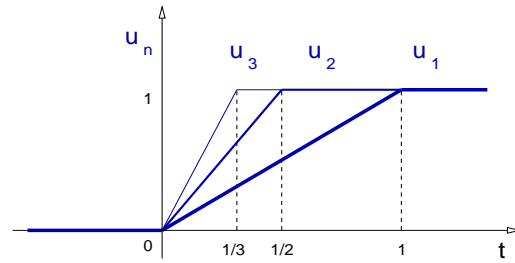
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Relation between deltas and steps.

Theorem

The sequence of functions for $n \geq 1$,

$$u_n(t) = \begin{cases} 0, & t < 0 \\ nt, & 0 \leq t \leq \frac{1}{n} \\ 1, & t > \frac{1}{n}. \end{cases}$$



satisfies, for $t \in (-\infty, 0) \cup (0, 1/n) \cup (1/n, \infty)$, both equations,

$$u'_n(t) = \delta_n(t), \quad \lim_{n \rightarrow \infty} u_n(t) = u(t), \quad t \in \mathbb{R}.$$

Remark:

- ▶ If we generalize the notion of derivative as $u'(t) = \lim_{n \rightarrow \infty} u'_n(t)$, then holds $u'(t) = \delta(t)$.
- ▶ Dirac's delta is a generalized derivative of the step function.

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Dirac's delta in Physics.

Remarks:

- (a) Dirac's delta generalized function is useful to describe *impulsive forces* in mechanical systems.
- (b) An impulsive force transmits a finite momentum in an infinitely short time.
- (c) For example: The momentum transmitted to a pendulum when hit by a hammer. Newton's law of motion says,

$$m v'(t) = F(t), \quad \text{with} \quad F(t) = F_0 \delta(t - t_0).$$

The momentum transfer is:

$$\Delta I = \lim_{\Delta t \rightarrow 0} m v(t) \Big|_{t_0 - \Delta t}^{t_0 + \Delta t} = \lim_{\Delta t \rightarrow 0} \int_{t_0 - \Delta t}^{t_0 + \Delta t} F(t) dt = F_0.$$

That is, $\Delta I = F_0$.

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The Laplace Transform of Dirac's delta.

Recall: The Laplace Transform can be generalized from functions to δ , as follows, $\mathcal{L}[\delta(t - c)] = \lim_{n \rightarrow \infty} \mathcal{L}[\delta_n(t - c)]$.

Theorem

$$\mathcal{L}[\delta(t - c)] = e^{-cs}.$$

Proof:

$$\mathcal{L}[\delta(t - c)] = \lim_{n \rightarrow \infty} \mathcal{L}[\delta_n(t - c)], \quad \delta_n(t) = n \left[u(t) - u\left(t - \frac{1}{n}\right) \right].$$

$$\mathcal{L}[\delta(t - c)] = \lim_{n \rightarrow \infty} n \left(\mathcal{L}[u(t - c)] - \mathcal{L}\left[u\left(t - c - \frac{1}{n}\right)\right] \right)$$

$$\mathcal{L}[\delta(t - c)] = \lim_{n \rightarrow \infty} n \left(\frac{e^{-cs}}{s} - \frac{e^{-(c+\frac{1}{n})s}}{s} \right) = e^{-cs} \lim_{n \rightarrow \infty} \frac{(1 - e^{-\frac{s}{n}})}{\left(\frac{s}{n}\right)}.$$

This is a singular limit, $\frac{0}{0}$. Use l'Hôpital rule.

The Laplace Transform of Dirac's delta.

Proof: Recall: $\mathcal{L}[\delta(t - c)] = e^{-cs} \lim_{n \rightarrow \infty} \frac{(1 - e^{-\frac{s}{n}})}{\left(\frac{s}{n}\right)}$.

$$\lim_{n \rightarrow \infty} \frac{(1 - e^{-\frac{s}{n}})}{\left(\frac{s}{n}\right)} = \lim_{n \rightarrow \infty} \frac{\left(-\frac{s}{n^2} e^{-\frac{s}{n}}\right)}{\left(-\frac{s}{n^2}\right)} = \lim_{n \rightarrow \infty} e^{-\frac{s}{n}} = 1.$$

We therefore conclude that $\mathcal{L}[\delta(t - c)] = e^{-cs}$. □

Remarks:

(a) This result is consistent with a previous result:

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) dt = f(t_0).$$

(b) $\mathcal{L}[\delta(t - c)] = \int_0^{\infty} \delta(t - c) e^{-st} dt = e^{-cs}$.

(c) $\mathcal{L}[\delta(t - c) f(t)] = \int_0^{\infty} \delta(t - c) e^{-st} f(t) dt = e^{-cs} f(c)$.

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Differential equations with Dirac's delta sources.

Example

Find the solution y to the initial value problem

$$y'' - y = -20 \delta(t - 3), \quad y(0) = 1, \quad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] - \mathcal{L}[y] = -20 \mathcal{L}[\delta(t - 3)]$.

$$\mathcal{L}[y''] = s^2 \mathcal{L}[y] - s y(0) - y'(0) \Rightarrow (s^2 - 1) \mathcal{L}[y] - s = -20 e^{-3s},$$

We arrive to the equation $\mathcal{L}[y] = \frac{s}{(s^2 - 1)} - 20 e^{-3s} \frac{1}{(s^2 - 1)}$,

$$\mathcal{L}[y] = \mathcal{L}[\cosh(t)] - 20 \mathcal{L}[u(t - 3) \sinh(t - 3)],$$

We conclude: $y(t) = \cosh(t) - 20 u(t - 3) \sinh(t - 3)$. \triangleleft

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[\delta(t - \pi)] - \mathcal{L}[\delta(t - 2\pi)]$,

$$(s^2 + 4)\mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s} \Rightarrow \mathcal{L}[y] = \frac{e^{-\pi s}}{(s^2 + 4)} - \frac{e^{-2\pi s}}{(s^2 + 4)},$$

$$\text{that is, } \mathcal{L}[y] = \frac{e^{-\pi s}}{2} \frac{2}{(s^2 + 4)} - \frac{e^{-2\pi s}}{2} \frac{2}{(s^2 + 4)}.$$

Recall: $e^{-cs} \mathcal{L}[f(t)] = \mathcal{L}[u(t - c)f(t - c)]$. Therefore,

$$\mathcal{L}[y] = \frac{1}{2} \mathcal{L}[u(t - \pi) \sin[2(t - \pi)]] - \frac{1}{2} \mathcal{L}[u(t - 2\pi) \sin[2(t - 2\pi)]].$$

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Solution: Recall:

$$\mathcal{L}[y] = \frac{1}{2} \mathcal{L}[u(t - \pi) \sin[2(t - \pi)]] - \frac{1}{2} \mathcal{L}[u(t - 2\pi) \sin[2(t - 2\pi)]].$$

This implies that,

$$y(t) = \frac{1}{2} u(t - \pi) \sin[2(t - \pi)] - \frac{1}{2} u(t - 2\pi) \sin[2(t - 2\pi)],$$

We conclude: $y(t) = \frac{1}{2} [u(t - \pi) - u(t - 2\pi)] \sin(2t)$. \triangleleft