

## Overview: Equations with singular points.

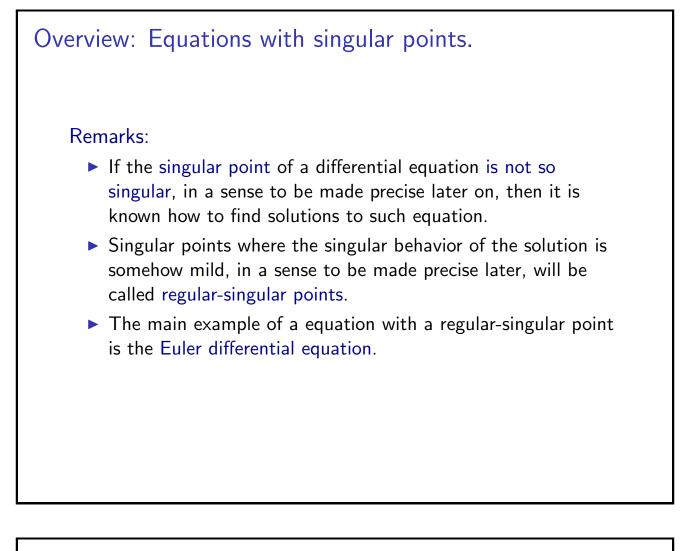
Recall: The point  $x_0 \in \mathbb{R}$  is a singular point of the equation

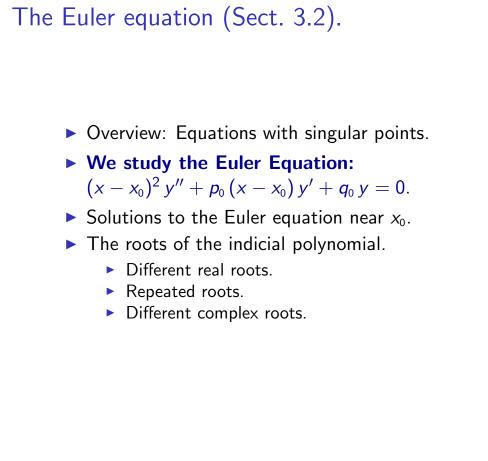
$$P(x) y'' + Q(x) y' + R(x) y = 0$$

iff holds  $P(x_0) = 0$ .

Remarks:

- We are interested in finding solutions to the equation above arbitrary close to a singular point x<sub>0</sub>.
- The order of the differential equation changes in a neighborhood of a singular point.
- In the limit  $x \to x_0$  the following could happen:
  - (a) The two linearly independent solutions remain bounded.
  - (b) Only one solution remains bounded.
  - (c) None solution remains bounded.





## The Euler equation

#### Definition

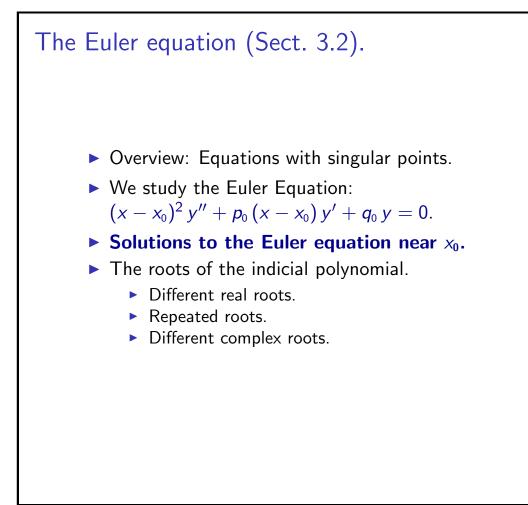
Given real constants  $p_0$ ,  $q_0$ , the *Euler differential equation* for the unknown y with singular point at  $x_0 \in R$  is given by

$$(x-x_0)^2 y'' + p_0 (x-x_0) y' + q_0 y = 0.$$

#### Remarks:

- ► The Euler equation has variable coefficients.
- Functions  $y(x) = e^{rx}$  are not solutions of the Euler equation.
- The point  $x_0 \in \mathbb{R}$  is a singular point of the equation.
- The particular case  $x_0 = 0$  is is given by

$$x^2 y'' + p_0 x y' + q_0 y = 0.$$



## Solutions to the Euler equation near $x_0$ . Summary of the main idea: • The main idea to find solution to the constant coefficients equation $y'' + a_1 y' + a_0 y = 0$ was to look for functions of the form $y(x) = e^{rx}$ . The exponential cancels out from the equation and we obtain an equation only for r without x, $(r^2 + a_1 r + a_0)e^{rx} = 0 \quad \Leftrightarrow \quad (r^2 + a_1 r + a_0) = 0.$ (1) • In the case of the Euler equation $x^2 y'' + p_0 x y' + q_0 y = 0$ the exponential functions $e^{rx}$ do not have the property given in Eq. (1), since $(x^2 r^2 + p_0 x r + q_0) e^{rx} = 0 \quad \Leftrightarrow \quad x^2 r^2 + p_0 x r + q_0 = 0,$ but the later equation still involves the variable x.

## Solutions to the Euler equation near $x_0$ .

Summary of the main idea: Look for solutions like  $y(x) = x^r$ . These function have the following property:

$$y'(x) = r x^{r-1} \quad \Rightarrow \quad x y'(x) = r x^r;$$
$$y''(x) = r(r-1) x^{r-2} \quad \Rightarrow \quad x^2 y''(x) = r(r-1) x^r.$$

Introduce  $y = x^r$  into Euler's equation  $x^2 y'' + p_0 x y' + q_0 y = 0$ , for  $x \neq 0$  we obtain

$$\left[r(r-1)+p_0r+q_0\right]x^r=0 \quad \Leftrightarrow \quad r(r-1)+p_0r+q_0=0.$$

The last equation involves only r, not x.

This equation is called the indicial equation, and is also called the Euler characteristic equation.

Solutions to the Euler equation near  $x_0$ . Theorem (Euler equation) Given  $p_0$ ,  $q_0$ ,  $x_0 \in \mathbb{R}$ , consider the Euler equation  $(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.$  (2) Let  $r_+$ ,  $r_-$  be solutions of  $r(r - 1) + p_0 r + q_0 = 0.$ (a) If  $r_+ \neq r_-$ , then a real-valued general solution of Eq. (2) is  $y(x) = c_0 |x - x_0|^{r_+} + c_1 |x - x_0|^{r_-}$ ,  $x \neq x_0$ ,  $c_0$ ,  $c_1 \in \mathbb{R}$ . (b) If  $r_+ = r_-$ , then a real-valued general solution of Eq. (2) is  $y(x) = \left[c_0 + c_1 \ln(|x - x_0|)\right] |x - x_0|^{r_+}$ ,  $x \neq x_0$ ,  $c_0$ ,  $c_1 \in \mathbb{R}$ . Given  $x_0 \neq x_1$ ,  $y_0$ ,  $y_1 \in \mathbb{R}$ , there is a unique solution to the IVP  $(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0$ ,  $y(x_1) = y_0$ ,  $y'(x_1) = y_1$ .

The Euler equation (Sect. 3.2).
Overview: Equations with singular points.
We study the Euler Equation: (x - x<sub>0</sub>)<sup>2</sup> y" + p<sub>0</sub> (x - x<sub>0</sub>) y' + q<sub>0</sub> y = 0.
Solutions to the Euler equation near x<sub>0</sub>.
The roots of the indicial polynomial.
Different real roots.
Repeated roots.
Different complex roots.

#### Different real roots.

#### Example

Find the general solution of the Euler equation

 $x^2 y'' + 4x y' + 2y = 0.$ 

Solution: We look for solutions of the form  $y(x) = x^r$ ,

$$x y'(x) = rx^r$$
,  $x^2 y''(x) = r(r-1)x^r$ .

Introduce  $y(x) = x^r$  into Euler equation,

$$\left[r(r-1)+4r+2\right]x^{r}=0 \quad \Leftrightarrow \quad r(r-1)+4r+2=0.$$

The solutions of  $r^2 + 3r + 2 = 0$  are given by

$$r_{\pm} = \frac{1}{2} \left[ -3 \pm \sqrt{9 - 8} \right] \quad \Rightarrow \quad r_{+} = -1 \qquad r_{-} = -2.$$

The general solution is  $y(x) = c_1 |x|^{-1} + c_2 |x|^{-2}$ .

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# The Euler equation (Sect. 3.2).

- Overview: Equations with singular points.
- We study the Euler Equation:  $(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.$
- Solutions to the Euler equation near x<sub>0</sub>.
- ► The roots of the indicial polynomial.
  - Different real roots.
  - Repeated roots.
  - Different complex roots.

### Repeated roots.

#### Example

Find the general solution of  $x^2 y'' - 3x y' + 4 y = 0$ . Solution: We look for solutions of the form  $y(x) = x^r$ ,

$$x y'(x) = rx^r$$
,  $x^2 y''(x) = r(r-1)x^r$ .

Introduce  $y(x) = x^r$  into Euler equation,

$$\left[r(r-1)-3r+4\right]x^{r}=0 \quad \Leftrightarrow \quad r(r-1)-3r+4=0.$$

The solutions of  $r^2 - 4r + 4 = 0$  are given by

$$r_{\pm} = \frac{1}{2} \left[ 4 \pm \sqrt{16 - 16} \right] \quad \Rightarrow \quad r_{+} = r_{-} = 2.$$

Two linearly independent solutions are

$$y_1(x) = x^2, \qquad y_2 = x^2 \ln(|x|).$$

The general solution is  $y(x) = c_1 x^2 + c_2 x^2 \ln(|x|)$ .

## The Euler equation (Sect. 3.2). • Overview: Equations with singular points. • We study the Euler Equation: $(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.$ • Solutions to the Euler equation near $x_0$ . • The roots of the indicial polynomial. • Different real roots. • Repeated roots. • Different complex roots.

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#### Different complex roots.

#### Example

Find the general solution of the Euler equation

$$x^2 y'' - 3x y' + 13 y = 0.$$

Solution: We look for solutions of the form  $y(x) = x^r$ ,

$$x y'(x) = rx^r$$
,  $x^2 y''(x) = r(r-1)x^r$ .

Introduce  $y(x) = x^r$  into Euler equation

$$[r(r-1)-3r+13]x^r = 0 \quad \Leftrightarrow \quad r(r-1)-3r+13 = 0.$$

The solutions of the indicial equation  $r^2 - 4r + 13 = 0$  are

$$r_{\pm} = \frac{1}{2} \left[ 4 \pm \sqrt{16 - 52} \right] \Rightarrow r_{\pm} = \frac{1}{2} \left[ 4 \pm \sqrt{-36} \right] \Rightarrow \begin{cases} r_{+} = 2 + 3i \\ r_{-} = 2 - 3i. \end{cases}$$

The general solution is  $y(x) = c_1 |x|^{(2+3i)} + c_2 |x|^{(2-3i)}$ .

#### Different complex roots.

Theorem (Real-valued fundamental solutions)

If  $p_0$ ,  $q_0 \in \mathbb{R}$  satisfy that  $[(p_0 - 1)^2 - 4q_0] < 0$ , then the indicial polynomial  $p(r) = r(r - 1) + p_0r + q_0$  of the Euler equation

$$x^{2} y'' + p_{0} x y' + q_{0} y = 0$$
 (3)

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has complex roots  $r_{+} = \alpha + i\beta$  and  $r_{-} = \alpha - i\beta$ , where

$$lpha = -rac{(p_{\scriptscriptstyle 0}-1)}{2}, \qquad eta = rac{1}{2} \sqrt{4q_{\scriptscriptstyle 0}-(p_{\scriptscriptstyle 0}-1)^2}.$$

Furthermore, a fundamental set of solution to Eq. (3) is

$$ilde{y}_1(x) = |x|^{(lpha+ieta)}, \qquad ilde{y}_2(x) = |x|^{(lpha-ieta)}$$

while another fundamental set of solutions to Eq. (3) is

$$y_1(x) = |x|^{\alpha} \cos\left(\beta \ln |x|\right), \qquad y_2(x) = |x|^{\alpha} \sin\left(\beta \ln |x|\right).$$

## Different complex roots.

**Proof**: Given 
$$\tilde{y}_1 = |x|^{(\alpha+i\beta)}$$
 and  $\tilde{y}_2 = |x|^{(\alpha-i\beta)}$ , introduce

$$y_1=rac{1}{2}ig( ilde y_1+ ilde y_2ig), \qquad y_1=rac{1}{2i}ig( ilde y_1- ilde y_2ig).$$

Use another Euler equation to rewrite  $\tilde{y}_1$  and  $\tilde{y}_2$ ,

$$\begin{split} \tilde{y}_1 &= |x|^{(\alpha+i\beta)} = |x|^{\alpha} \, |x|^{i\beta} = |x|^{\alpha} \, e^{\ln(|x|^{i\beta})} = |x|^{\alpha} \, e^{i\beta \ln(|x|)}.\\ \tilde{y}_1 &= |x|^{\alpha} \big[ \cos\big(\beta \ln |x|\big) + 1 \sin\big(\beta \ln |x|\big) \big],\\ \tilde{y}_2 &= |x|^{\alpha} \big[ \cos\big(\beta \ln |x|\big) - 1 \sin\big(\beta \ln |x|\big) \big]. \end{split}$$

We conclude that

$$y_1(x) = |x|^{\alpha} \cos(\beta \ln |x|), \qquad y_2(x) = |x|^{\alpha} \sin(\beta \ln |x|).$$

## Different complex roots.

#### Example

Find a real-valued general solution of the Euler equation

$$x^2 y'' - 3x y' + 13 y = 0.$$

Solution: The indicial equation is r(r-1) - 3r + 13 = 0. The solutions of the indicial equations are

 $r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r_+ = 2 + 3i, \quad r_- = 2 - 3i.$ 

A complex-valued general solution is

$$y(x) = \tilde{c}_1 |x|^{(2+3i)} + \tilde{c}_2 |x|^{(2-3i)} \quad \tilde{c}_1, \, \, \tilde{c}_2 \in \mathbb{C}.$$

A real-valued general solution is

 $y(x) = c_1 |x|^2 \cos(3 \ln |x|) + c_2 |x|^2 \sin(3 \ln |x|), \quad c_1, \ c_2 \in \mathbb{R}.$