Second order linear ODE (Sect. 2.2).

- Review: Second order linear differential equations.
- Idea: Soving constant coefficients equations.
- The characteristic equation.
- Solution formulas for constant coefficients equations.

Review: Second order linear ODE.

Definition

Given functions a_1, a_0, $b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t)$$

is called a *second order linear* differential equation. If $b = 0$, the equation is called *homogeneous*. If the coefficients a_1, $a_2 \in \mathbb{R}$ are constants, the equation is called of *constant coefficients*.

Theorem (Superposition property)

If the functions y_1 and y_2 are solutions to the homogeneous linear equation

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0,$$

then the linear combination $c_1 y_1(t) + c_2 y_2(t)$ is also a solution for any constants c_1, $c_2 \in \mathbb{R}$.

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Idea: Soving constant coefficients equations.

Remark: Just by trial and error one can find solutions to second order, constant coefficients, homogeneous, linear differential equations. We present the main ideas with an example.

Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: We look for solutions proportional to exponentials e^{rt}, for an appropriate constant $r \in \mathbb{R}$, since the exponential can be canceled out from the equation.
If $y(t) = e^{rt}$, then $y'(t) = re^{rt}$, and $y''(t) = r^2 e^{rt}$. Hence

$$(r^2 + 5r + 6)e^{rt} = 0 \quad \Leftrightarrow \quad r^2 + 5r + 6 = 0.$$

That is, r must be a root of the polynomial $p(r) = r^2 + 5r + 6$.

This polynomial is called the characteristic polynomial of the differential equation.

Idea: Soving constant coefficients equations.

Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: Recall: $p(r) = r^2 + 5r + 6$.

The roots of the characteristic polynomial are

$$r = \frac{1}{2}\left(-5 \pm \sqrt{25 - 24}\right) = \frac{1}{2}\left(-5 \pm 1\right) \quad \Rightarrow \quad \begin{cases} r_1 = -2, \\ r_2 = -3. \end{cases}$$

Therefore, we have found two solutions to the ODE,

$$y_1(t) = e^{-2t}, \qquad y_2(t) = e^{-3t}.$$

Their superposition provides infinitely many solutions,

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}. \qquad \triangleleft$$

Idea: Soving constant coefficients equations.

Summary: The differential equation $y'' + 5y' + 6y = 0$ has infinitely many solutions,

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Remarks:

- There are two free constants in the solution found above.
- The ODE above is second order, so two integrations must be done to find the solution. This explain the origin of the two free constant in the solution.
- An IVP for a second order differential equation will have a unique solution if the IVP contains two initial conditions.

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The characteristic equation.

Definition

Given a second order linear homogeneous differential equation with constant coefficients

$$y'' + a_1 y' + a_0 = 0, \qquad (1)$$

the *characteristic polynomial* and the *characteristic equation* associated with the differential equation in (1) are, respectively,

$$p(r) = r^2 + a_1 r + a_0, \qquad p(r) = 0.$$

Remark: If r_1, r_2 are the solutions of the characteristic equation and c_1, c_2 are constants, then we will show that the general solution of Eq. (1) is given by

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

The characteristic equation.

Example

Find the solution y of the initial value problem

$$y'' + 5y' + 6 = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

Solution: A solution of the differential equation above is

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

We now find the constants c_1 and c_2 that satisfy the initial conditions above:

$$1 = y(0) = c_1 + c_2, \qquad -1 = y'(0) = -2c_1 - 3c_2.$$

$$c_1 = 1 - c_2 \Rightarrow 1 = 2(1 - c_2) + 3c_2 \Rightarrow c_2 = -1 \Rightarrow c_1 = 2.$$

Therefore, the unique solution to the initial value problem is

$$y(t) = 2e^{-2t} - e^{-3t}.$$

$\triangleleft$

The characteristic equation.

Example

Find the general solution y of the differential equation

$$2y'' - 3y' + y = 0.$$

Solution: We look for every solution of the form $y(t) = e^{rt}$, where r is a solution of the characteristic equation

$$2r^2 - 3r + 1 = 0 \;\Rightarrow\; r = \frac{1}{4}\left(3 \pm \sqrt{9 - 8}\right) \;\Rightarrow\; \begin{cases} r_1 = 1, \\ r_2 = \dfrac{1}{2}. \end{cases}$$

Therefore, the general solution of the equation above is

$$y(t) = c_1 e^{t} + c_2 e^{t/2},$$

where c_1, c_2 are arbitrary constants.

$\triangleleft$

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Solution formulas for constant coefficients equations.

Theorem (Constant coefficients)

Given real constants a_1, a_0, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1\, y' + a_0\, y = 0.$$

Let r_+, r_- be the roots of the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$, and let c_0, c_1 be arbitrary constants. Then, the general solution of the differential eqation is given by:

(a) *If $r_+ \neq r_-$, real or complex, then $y(t) = c_0\, e^{r_+ t} + c_1\, e^{r_- t}$.*

(b) *If $r_+ = r_- = \hat{r} \in \mathbb{R}$, then is $y(t) = c_0\, e^{\hat{r} t} + c_1\, t e^{\hat{r} t}$.*

Furthermore, given real constants t_0, y_0 and y_1, there is a unique solution to the initial value problem

$$y'' + a_1\, y' + a_0\, y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_1.$$