

Modeling with first order equations (Sect. 1.5).

- ▶ Radioactive decay.
 - ▶ Carbon-14 dating.
- ▶ Salt in a water tank.
 - ▶ The experimental device.
 - ▶ The main equations.
 - ▶ Analysis of the mathematical model.
 - ▶ Predictions for particular situations.

Radioactive decay

Remarks:

- (a) Radioactive substances randomly emit protons, electrons, radiation, and they are transformed in another substance.
- (b) It can be seen that the time rate of change of the amount N of a radioactive substance is proportional to the negative amount of radioactive substance.

$$N'(t) = -a N(t), \quad N(0) = N_0, \quad a > 0.$$

- (c) The integrating factor method implies $N(t) = N_0 e^{-at}$.
- (d) The *half-life* is the time τ needed to get $N(\tau) = N_0/2$.

$$N_0 e^{-a\tau} = \frac{N_0}{2} \Rightarrow -a\tau = \ln\left(\frac{1}{2}\right) \Rightarrow \tau = \frac{\ln(2)}{a}.$$

- (e) Using the half-life, we get $N(t) = N_0 2^{-t/\tau}$.

Radioactive decay

Example

Remains containing 14% of the original amount of Carbon-14 are found. Knowing that Carbon-14 half-life is $\tau = 5730$ years, date the remains.

Solution: Set $t = 0$ when the organism dies. Since the amount N of Carbon-14 only decays after the organism dies,

$$N(t) = N_0 2^{-t/\tau}, \quad \tau = 5730 \text{ years.}$$

The remains contain 14% of the original amount at the time t ,

$$\frac{N(t)}{N_0} = \frac{14}{100} \Rightarrow 2^{-t/\tau} = \frac{14}{100}$$

$$-\frac{t}{\tau} = \log_2(14/100) \Rightarrow t = \tau \log_2(100/14).$$

The organism died 16,253 years ago.



Modeling with first order equations (Sect. 1.5).

- ▶ Radioactive decay.
 - ▶ Carbon-14 dating.
- ▶ **Salt in a water tank.**
 - ▶ The experimental device.
 - ▶ The main equations.
 - ▶ Analysis of the mathematical model.
 - ▶ Predictions for particular situations.

Salt in a water tank.

Problem: Describe the salt concentration in a tank with water if salty water comes in and goes out of the tank.

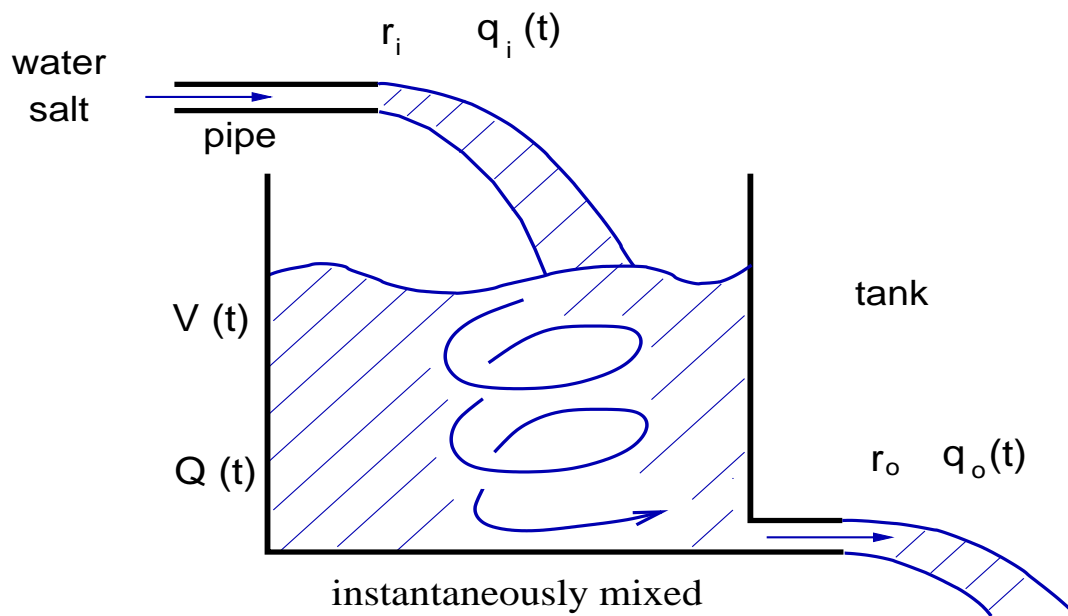
Main ideas of the test:

- ▶ Since the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
- ▶ The amount of salt in the tank depends on the salt concentration coming in and going out of the tank.
- ▶ The salt in the tank also depends on the water rates coming in and going out of the tank.
- ▶ To construct a model means to find the differential equation that takes into account the above properties of the system.
- ▶ Finding the solution to the differential equation with a particular initial condition means we can predict the evolution of the salt in the tank if we know the tank initial condition.

Modeling with first order equations (Sect. 1.5).

- ▶ Radioactive decay.
 - ▶ Carbon-14 dating.
- ▶ **Main example: Salt in a water tank.**
 - ▶ **The experimental device.**
 - ▶ The main equations.
 - ▶ Analysis of the mathematical model.
 - ▶ Predictions for particular situations.

The experimental device.



The experimental device.

Definitions:

- ▶ $r_i(t)$, $r_o(t)$: Rates in and out of water entering and leaving the tank at the time t .
- ▶ $q_i(t)$, $q_o(t)$: Salt concentration of the water entering and leaving the tank at the time t .
- ▶ $V(t)$: Water volume in the tank at the time t .
- ▶ $Q(t)$: Salt mass in the tank at the time t .

Units:

$$[r_i(t)] = [r_o(t)] = \frac{\text{Volume}}{\text{Time}}, \quad [q_i(t)] = [q_o(t)] = \frac{\text{Mass}}{\text{Volume}}.$$

$$[V(t)] = \text{Volume}, \quad [Q(t)] = \text{Mass}.$$

Modeling with first order equations (Sect. 1.5).

- ▶ Radioactive decay.
 - ▶ Carbon-14 dating.
- ▶ **Main example: Salt in a water tank.**
 - ▶ The experimental device.
 - ▶ **The main equations.**
 - ▶ Analysis of the mathematical model.
 - ▶ Predictions for particular situations.

The main equations.

Remark: The mass conservation provides the main equations of the mathematical description for salt in water.

Main equations:

$$\frac{d}{dt}V(t) = r_i(t) - r_o(t), \quad \text{Volume conservation,} \quad (1)$$

$$\frac{d}{dt}Q(t) = r_i(t)q_i(t) - r_o(t)q_o(t), \quad \text{Mass conservation,} \quad (2)$$

$$q_o(t) = \frac{Q(t)}{V(t)}, \quad \text{Instantaneously mixed,} \quad (3)$$

$$r_i, r_o : \quad \text{Constants.} \quad (4)$$

The main equations.

Remarks:

$$\left[\frac{dV}{dt} \right] = \frac{\text{Volume}}{\text{Time}} = [r_i - r_o],$$

$$\left[\frac{dQ}{dt} \right] = \frac{\text{Mass}}{\text{Time}} = [r_i q_i - r_o q_o],$$

$$[r_i q_i - r_o q_o] = \frac{\text{Volume}}{\text{Time}} \frac{\text{Mass}}{\text{Volume}} = \frac{\text{Mass}}{\text{Time}}.$$

Modeling with first order equations (Sect. 1.5).

- ▶ Radioactive decay.
 - ▶ Carbon-14 dating.
- ▶ **Main example: Salt in a water tank.**
 - ▶ The experimental device.
 - ▶ The main equations.
 - ▶ **Analysis of the mathematical model.**
 - ▶ Predictions for particular situations.

Analysis of the mathematical model.

Eqs. (??) and (??) imply

$$V(t) = (r_i - r_o) t + V_0, \quad (5)$$

where $V(0) = V_0$ is the initial volume of water in the tank.

Eqs. (??) and (??) imply

$$\frac{d}{dt} Q(t) = r_i q_i(t) - r_o \frac{Q(t)}{V(t)}. \quad (6)$$

Eqs. (??) and (??) imply

$$\frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \quad (7)$$

Analysis of the mathematical model.

Recall:
$$\frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t).$$

Notation:
$$a(t) = -\frac{r_o}{(r_i - r_o) t + V_0}, \text{ and } b(t) = r_i q_i(t).$$

The main equation of the description is given by

$$Q'(t) = a(t) Q(t) + b(t).$$

Linear ODE for Q . Solution: Integrating factor method.

$$Q(t) = e^{A(t)} \left[Q_0 + \int_0^t e^{-A(s)} b(s) ds \right]$$

with $Q(0) = Q_0$, and
$$A(t) = \int_0^t a(s) ds.$$

Modeling with first order equations (Sect. 1.5).

- ▶ Radioactive decay.
 - ▶ Carbon-14 dating.
- ▶ **Main example: Salt in a water tank.**
 - ▶ The experimental device.
 - ▶ The main equations.
 - ▶ Analysis of the mathematical model.
 - ▶ **Predictions for particular situations.**

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and q_i are constants.

If r , q_i , Q_0 and V_0 are given, find $Q(t)$.

Solution: Always holds $Q'(t) = a(t)Q(t) + b(t)$.

In this case:

$$a(t) = -\frac{r_o}{(r_i - r_o)t + V_0} \Rightarrow a(t) = -\frac{r}{V_0} = -a_0,$$

$$b(t) = r_i q_i(t) \Rightarrow b(t) = r q_i = b_0.$$

We need to solve the IVP:

$$Q'(t) = -a_0 Q(t) + b_0, \quad Q(0) = Q_0.$$

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and q_i are constants.
If r , q_i , Q_0 and V_0 are given, find $Q(t)$.

Solution: Recall the IVP: $Q'(t) + a_0 Q(t) = b_0$, $Q(0) = Q_0$.

Integrating factor method:

$$A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad e^{a_0 t} Q(t) = Q_0 + \int_0^t e^{a_0 s} b_0 ds.$$

$$Q(t) = e^{-a_0 t} \left[Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right] = \left(Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0}.$$

But $\frac{b_0}{a_0} = r q_i \frac{V_0}{r} = q_i V_0$, and $a_0 = \frac{r}{V_0}$. We conclude:

$$Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0.$$

Predictions for particular situations.

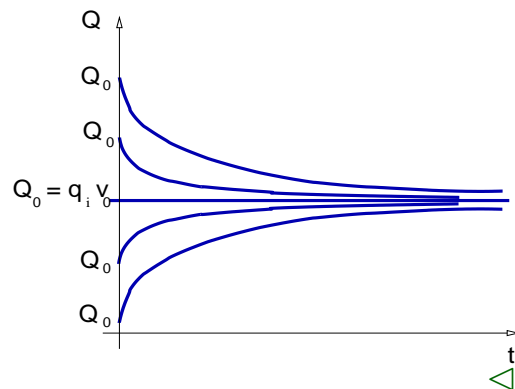
Example

Assume that $r_i = r_o = r$ and q_i are constants.
If r , q_i , Q_0 and V_0 are given, find $Q(t)$.

Solution: Recall: $Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0$.

Particular cases:

- ▶ $\frac{Q_0}{V_0} > q_i$;
- ▶ $\frac{Q_0}{V_0} = q_i$, so $Q(t) = Q_0$;
- ▶ $\frac{Q_0}{V_0} < q_i$.



Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and q_i are constants.

If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0$, we get

$$Q(t) = Q_0 e^{-rt/V_0}.$$

Since $V(t) = (r_i - r_o) t + V_0$ and $r_i = r_o$, we obtain $V(t) = V_0$.

So $q(t) = Q(t)/V(t)$ is given by $q(t) = \frac{Q_0}{V_0} e^{-rt/V_0}$. Therefore,

$$\frac{1}{100} \frac{Q_0}{V_0} = q(t_1) = \frac{Q_0}{V_0} e^{-rt_1/V_0} \Rightarrow e^{-rt_1/V_0} = \frac{1}{100}.$$

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and q_i are constants.

If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \frac{1}{100}$. Then,

$$-\frac{r}{V_0} t_1 = \ln\left(\frac{1}{100}\right) = -\ln(100) \Rightarrow \frac{r}{V_0} t_1 = \ln(100).$$

We conclude that $t_1 = \frac{V_0}{r} \ln(100)$.

In this case: $t_1 = 100 \ln(100)$.

◁

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ are constants. If $r = 5 \times 10^6$ gal/year, $q_i(t) = 2 + \sin(2t)$ grams/gal, $V_0 = 10^6$ gal, $Q_0 = 0$, find $Q(t)$.

Solution: Recall: $Q'(t) = a(t)Q(t) + b(t)$. In this case:

$$a(t) = -\frac{r_o}{(r_i - r_o)t + V_0} \Rightarrow a(t) = -\frac{r}{V_0} = -a_0,$$

$$b(t) = r_i q_i(t) \Rightarrow b(t) = r[2 + \sin(2t)].$$

We need to solve the IVP: $Q'(t) = -a_0 Q(t) + b(t)$, $Q(0) = 0$.

$$e^{a_0 t} Q(t) = \int_0^t e^{a_0 s} b(s) ds.$$

We conclude: $Q(t) = re^{-rt/V_0} \int_0^t e^{rs/V_0} [2 + \sin(2s)] ds$.