Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: Eigenvalue-eigenfunction problem.

Two-point Boundary Value Problem.

Definition

A *two-point BVP* is the following: Given functions p, q, g, and constants
$$x_1 < x_2, \quad y_1, y_2, \quad b_1, b_2, \quad \tilde{b}_1, \tilde{b}_2,$$
find a function y solution of the differential equation
$$y'' + p(x)\, y' + q(x)\, y = g(x),$$
together with the extra, *boundary conditions*,
$$b_1\, y(x_1) + b_2\, y'(x_1) = y_1,$$
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Remarks:

- Both y and y' might appear in the boundary condition, evaluated at the same point.
- In this notes we only study the case of constant coefficients,

$$y'' + a_1\, y' + a_0\, y = g(x).$$

Two-point Boundary Value Problem.

Example

Examples of BVP.

Two-point Boundary Value Problem.

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Examples of BVP. Assume $x_1 \neq x_2$.

(1) Find y solution of

$$y'' + a_1\, y' + a_0\, y = g(x), \quad y(x_1) = y_1, \quad y(x_2) = y_2.$$

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Problem: The equilibrium (time independent) temperature of a bar of length L with insulated horizontal sides and the bar vertical extremes kept at fixed temperatures T_0, T_L is the solution of the BVP:

$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$

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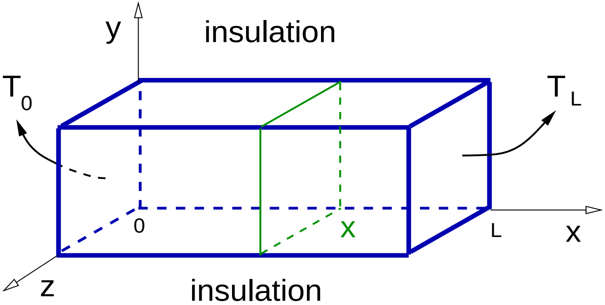

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Comparison: IVP vs BVP.

Review: IVP:

Find the function values $y(t)$ solutions of the differential equation

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Remark: In physics:

- $y(t)$: Position at time t.

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Remark: In physics:

- $y(t)$: Position at time t.
- Initial conditions: Position and velocity at the initial time t_0.

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Remark: In physics:

- $y(x)$: A physical quantity (temperature) at a position x.
- Boundary conditions: Conditions at the boundary of the object under study, where $x_1 \neq x_2$.

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Existence, uniqueness of solutions to BVP.

Review: The initial value problem.

Theorem (IVP)

Consider the homogeneous initial value problem:

$$y'' + a_1\, y' + a_0\, y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_1,$$

and let $r_\pm$ be the roots of the characteristic polynomial

$$p(r) = r^2 + a_1\, r + a_0.$$

If $r_+ \neq r_-$, real or complex, then for every choice of y_0, y_1, there exists a unique solution y to the initial value problem above.

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If $r_+ \neq r_-$, real or complex, then for every choice of y_0, y_1, there exists a unique solution y to the initial value problem above.

Summary: The IVP above always has a unique solution, no matter what y_0 and y_1 we choose.

Existence, uniqueness of solutions to BVP.

Theorem (BVP)

Consider the homogeneous boundary value problem:

$$y'' + a_1\, y' + a_0\, y = 0, \qquad y(0) = y_0, \quad y(L) = y_1,$$

and let $r_\pm$ *be the roots of the characteristic polynomial*

$$p(r) = r^2 + a_1\, r + a_0.$$

(A) *If* $r_+ \neq r_-$*, real, then for every choice of* $L \neq 0$ *and* y_0*,* y_1*, there exists a unique solution* y *to the BVP above.*

(B) *If* $r_\pm = \alpha \pm i\beta$*, with* $\beta \neq 0$*, and* $\alpha, \beta \in \mathbb{R}$*, then the solutions to the BVP above belong to one of these possibilities:*

  (1) *There exists a unique solution.*
  (2) *There exists no solution.*
  (3) *There exist infinitely many solutions.*

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$.

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$$y(t) = c_1\, e^{r_- t} + c_2\, e^{r_+ t},$$

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Using matrix notation,

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The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the $\det(Z) \neq 0$,

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$$Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_-\, e^{r_- t_0} & r_+\, e^{r_+ t_0} \end{bmatrix}$$

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Proof of IVP:

Recall: $Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

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A simple calculation shows

$$\det(Z) = \left(r_+ - r_-\right) e^{(r_+ + r_-)\, t_0}$$

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Since $r_+ \neq r_-$, the matrix Z is invertible

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We conclude that for every choice of y_0 and y_1, there exist a unique value of c_1 and c_2, so the IVP above has a unique solution. $\square$

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Using matrix notation,

$$\begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the $\det(Z) \neq 0$, where

$$Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix}$$

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1\, e^{r_- x} + c_2\, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine c_1 and c_2 as follows:

$$y_0 = y(0) = c_1 + c_2.$$

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Existence, uniqueness of solutions to BVP.

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Existence, uniqueness of solutions to BVP.

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(1) If $\beta L \neq n\pi$, then BVP has a unique solution.

(2) If $\beta L = n\pi$ then BVP either has no solutions or it has infinitely many solutions. $\square$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

Existence, uniqueness of solutions to BVP.

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The BVP has a unique solution. $\triangleleft$

Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: **Eigenvalue-eigenfunction problem.**

Particular case of BVP: Eigenvalue-eigenfunction problem.

Problem:

Find a number λ and a non-zero function y solutions to the boundary value problem

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

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Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

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$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right),$$

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Remarks: We will show that:

(1) If $\lambda \leqslant 0$, then the BVP has no solution.

(2) If $\lambda > 0$, then there exist infinitely many eigenvalues λ_n and eigenfunctions y_n, with n any positive integer, given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right),$$

(3) Analogous results can be proven for the same equation but with different types of boundary conditions.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

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The boundary conditions imply

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$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x.$$

The boundary conditions imply

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Since $y = 0$, there are NO non-zero solutions for $\lambda = 0$.

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Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

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Solution: Case $\lambda < 0$. Introduce the notation $\lambda = -\mu^2$.

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$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

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We need to solve the linear system

$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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Since $\det(Z) = e^{-\mu L} - e^{\mu L} \neq 0$ for $L \neq 0$, matrix Z is invertible, so the linear system above has a unique solution $c_1 = 0$ and $c_2 = 0$.

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Solution: Case $\lambda > 0$.

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The general solution is

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Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

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$$0 = y(L) = c_2 \sin(\mu L), \quad c_2 \neq 0$$

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Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

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Solution: Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

The non-zero solution condition is the reason for $c_2 \neq 0$.

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Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

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$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi$$

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$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L}.$$

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The non-zero solution condition is the reason for $c_2 \neq 0$. Hence

$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L}.$$

Recalling that $\lambda_n = \mu_n^2$, and choosing $c_2 = 1$,

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Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

The non-zero solution condition is the reason for $c_2 \neq 0$. Hence

$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L}.$$

Recalling that $\lambda_n = \mu_n^2$, and choosing $c_2 = 1$, we conclude

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right). \qquad \triangleleft$$

Overview of Fourier Series (Sect. 6.2).

- Origins of the Fourier Series.
- Periodic functions.
- Orthogonality of Sines and Cosines.
- Main result on Fourier Series.

Origins of the Fourier Series.

Summary:

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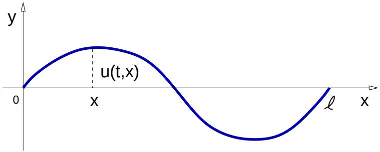

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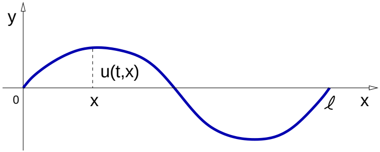

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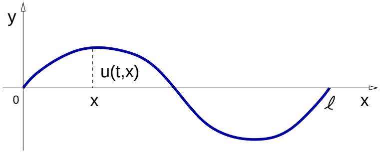

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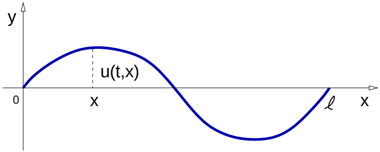

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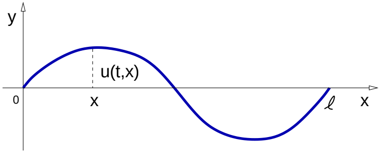

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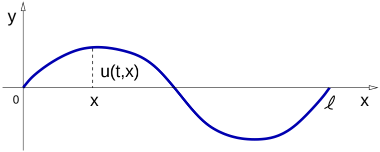

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Remark: The wave equation and its solutions provide a mathematical description of music.

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- More precisely: Every continuous, τ-periodic function F, there exist constants a_0, a_n, b_n, for $n = 1, 2, \cdots$ such that

$$F_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \Big[a_n \cos\Big(\frac{2n\pi x}{\tau}\Big) + b_n \sin\Big(\frac{2n\pi x}{\tau}\Big) \Big],$$

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Notation: $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{2n\pi x}{\tau}\Big) + b_n \sin\Big(\frac{2n\pi x}{\tau}\Big) \Big].$

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Given a continuous, τ-periodic function f, find the formulas for a_n and b_n such that

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Remarks: We need to review two main concepts:

- The notion of periodic functions.
- The notion of orthogonal functions, in particular the orthogonality of Sines and Cosines.

Fourier Series (Sect. 6.2).

- Origins of the Fourier Series.
- **Periodic functions.**
- Orthogonality of Sines and Cosines.
- Main result on Fourier Series.

Periodic functions.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* iff there exists $\tau > 0$ such that for all $x \in \mathbb{R}$ holds

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Notation:

A periodic function with period T is also called T-periodic.

Periodic functions.

Example

The following functions are periodic, with period T,

$$f(x) = \sin(x), \qquad T = 2\pi.$$

$$f(x) = \cos(x), \qquad T = 2\pi.$$

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$$f(x) = \sin(ax), \qquad T = \frac{2\pi}{a}.$$

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Periodic functions.

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The following functions are periodic, with period T,

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$\triangleleft$

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Show that the function below is periodic, and find its period,

$$f(x) = e^x, \qquad x \in [0, 2), \qquad f(x - 2) = f(x).$$

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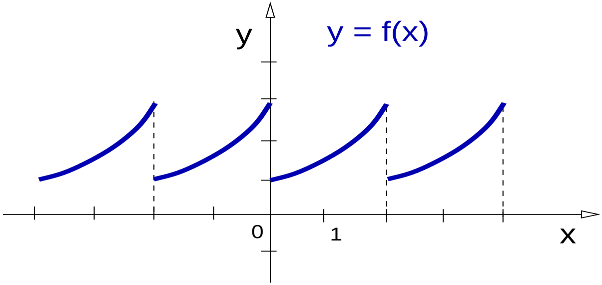

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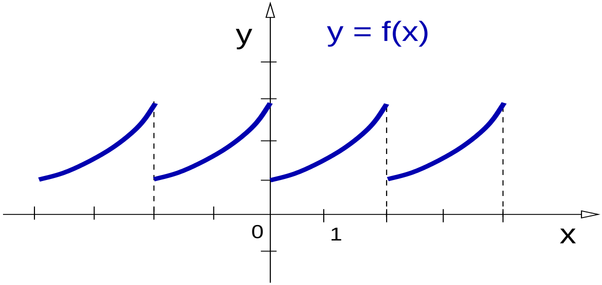

So the function is periodic with period $T = 2$. $\triangleleft$

Periodic functions.

Theorem

A linear combination of T-periodic functions is also T-periodic.

Periodic functions.

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Proof: If $f(x+T) = f(x)$ and $g(x+T) = g(x)$, then

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so $(af + bg)$ is also T-periodic. $\square$

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Since f and g are invariant under translations by τ/n, they are also invariant under translations by τ.

Periodic functions.

Corollary

Any function f given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi x}{\tau}\right) + b_n \sin\left(\frac{2n\pi x}{\tau}\right) \right]$$

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Remark: We will show that the converse statement is true.

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Theorem

A function f is τ-periodic iff holds

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{2n\pi x}{\tau}\Big) + b_n \sin\Big(\frac{2n\pi x}{\tau}\Big) \Big].$$

Fourier Series (Sect. 6.2).

- Origins of the Fourier Series.
- Periodic functions.
- **Orthogonality of Sines and Cosines.**
- Main result on Fourier Series.

Orthogonality of Sines and Cosines.

Remark:

From now on we work on the following domain: $[-L, L]$.

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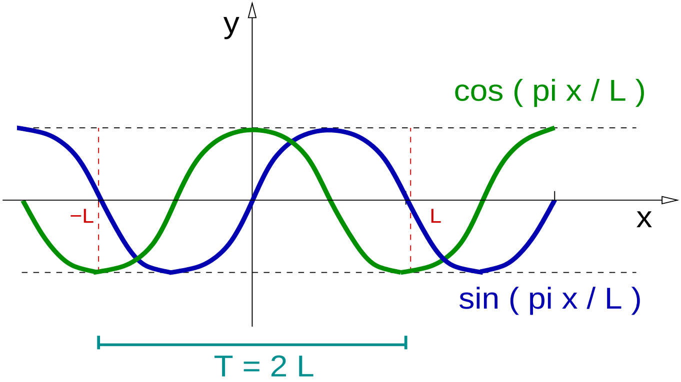

Orthogonality of Sines and Cosines.

Theorem (Orthogonality)

The following relations hold for all $n, m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

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Remark:

- The operation $f \cdot g = \displaystyle\int_{-L}^{L} f(x)\,g(x)\,dx$ is an inner product in the vector space of functions.

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- The operation $f \cdot g = \displaystyle\int_{-L}^{L} f(x)\,g(x)\,dx$ is an inner product in the vector space of functions. Like the dot product is in $\mathbb{R}^2$.
- Two functions f, g, are orthogonal iff $f \cdot g = 0$.

Orthogonality of Sines and Cosines.

Recall:

$$\cos(\theta)\,\cos(\phi) = \frac{1}{2}\big[\cos(\theta+\phi) + \cos(\theta-\phi)\big];$$

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Proof: First formula: If $n = m = 0$, it is simple to see that

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In the case where one of n or m is non-zero, use the relation

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx$$
$$+\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

Orthogonality of Sines and Cosines.

Proof: Since one of n or m is non-zero,

Orthogonality of Sines and Cosines.

Proof: Since one of n or m is non-zero, holds

$$\frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx = \frac{L}{2(n+m)\pi}\,\sin\Big[\frac{(n+m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

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We obtain that

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This establishes the first equation in the Theorem. The remaining equations are proven in a similar way. $\square$

Overview of Fourier Series (Sect. 6.2).

- Origins of the Fourier Series.
- Periodic functions.
- Orthogonality of Sines and Cosines.
- **Main result on Fourier Series.**

Main result on Fourier Series.

Theorem (Fourier Series)

If the function $f : [-L, L] \subset \mathbb{R} \to \mathbb{R}$ is continuous, then f can be expressed as an infinite series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big] \qquad (1)$$

with the constants a_n and b_n given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

Furthermore, the Fourier series in Eq. (1) provides a $2L$-periodic extension of f from the domain $[-L, L] \subset \mathbb{R}$ to $\mathbb{R}$.

Examples of the Fourier Theorem (Sect. 6.2).

- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.

The Fourier Theorem: Continuous case.

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Furthermore, the Fourier series in Eq. (2) provides a $2L$-periodic extension of function f from the domain $[-L, L] \subset \mathbb{R}$ to $\mathbb{R}$.

The Fourier Theorem: Continuous case.

Sketch of the Proof:

- Define the partial sum functions

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big]$$

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- Express f_N as a convolution of Sine, Cosine, functions and the original function f.

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- Express f_N as a convolution of Sine, Cosine, functions and the original function f.
- Use the convolution properties to show that

$$\lim_{N\to\infty} f_N(x) = f(x), \qquad x \in [-L, L]. \qquad \square$$

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Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

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$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (1+x)\,dx + \int_{0}^{1} (1-x)\,dx.$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: In this case $L = 1$. The Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x)\big],$$

where the a_n, b_n are given in the Theorem. We start with a_0,

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (1 + x)\,dx + \int_{0}^{1} (1 - x)\,dx.$$

$$a_0 = \Big(x + \frac{x^2}{2}\Big)\Big|_{-1}^{0} + \Big(x - \frac{x^2}{2}\Big)\Big|_{0}^{1}$$

Example: Using the Fourier Theorem.

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Example: Using the Fourier Theorem.

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We obtain: $a_0 = 1$.

Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$.

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$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$. Similarly, the rest of the a_n are given by,

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx$$

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Recall the integrals $\displaystyle\int \cos(n\pi x)\, dx = \frac{1}{n\pi} \sin(n\pi x),$

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Recall the integrals $\displaystyle\int \cos(n\pi x)\, dx = \frac{1}{n\pi} \sin(n\pi x)$, and

$$\int x \cos(n\pi x)\, dx = \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x).$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: It is not difficult to see that

$$a_n = \frac{1}{n\pi}\,\sin(n\pi x)\Big|_{-1}^{0} + \Big[\frac{x}{n\pi}\,\sin(n\pi x) + \frac{1}{n^2\pi^2}\,\cos(n\pi x)\Big]\Big|_{-1}^{0}$$
$$+\frac{1}{n\pi}\,\sin(n\pi x)\Big|_{0}^{1} - \Big[\frac{x}{n\pi}\,\sin(n\pi x) + \frac{1}{n^2\pi^2}\,\cos(n\pi x)\Big]\Big|_{0}^{1}$$

Example: Using the Fourier Theorem.

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$$a_n = \Big[\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} \cos(-n\pi)\Big] - \Big[\frac{1}{n^2\pi^2} \cos(n\pi) - \frac{1}{n^2\pi^2}\Big].$$

Example: Using the Fourier Theorem.

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We then conclude that $a_n = \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$, and $a_n = \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]$.

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Finally, we must find the coefficients b_n.

Example: Using the Fourier Theorem.

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Example: Using the Fourier Theorem.

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Finally, we must find the coefficients b_n.

A similar calculation shows that $b_n = 0$.

Then, the Fourier series of f is given by

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big] \cos(n\pi x).$$

$\triangleleft$

Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

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We can obtain a simpler expression for the Fourier coefficients a_n.

Example: Using the Fourier Theorem.

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Recall the relations $\cos(n\pi) = (-1)^n$,

Example: Using the Fourier Theorem.

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$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \big[1 + (-1)^{n+1}\big] \cos(n\pi x).$$

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If $n = 2k$,

Example: Using the Fourier Theorem.

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If $n = 2k$, so n is even,

Example: Using the Fourier Theorem.

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If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd,

Example: Using the Fourier Theorem.

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If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2 \pi^2} (1 - 1)$$

Example: Using the Fourier Theorem.

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If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2\pi^2}(1 - 1) \quad \Rightarrow \quad a_{2k} = 0.$$

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If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

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If $n = 2k - 1$, so n is odd, so $n + 1 = 2k$ is even, then

$$a_{2k-1} = \frac{2}{(2k-1)^2\pi^2}(1 + 1)$$

Example: Using the Fourier Theorem.

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If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

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If $n = 2k - 1$, so n is odd, so $n + 1 = 2k$ is even, then

$$a_{2k-1} = \frac{2}{(2k-1)^2\pi^2} (1 + 1) \quad \Rightarrow \quad a_{2k-1} = \frac{4}{(2k-1)^2\pi^2}.$$

Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

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Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1 + (-1)^{n+1}\big] \cos(n\pi x)$, and

$$a_{2k} = 0, \qquad a_{2k-1} = \frac{4}{(2k-1)^2\pi^2}.$$

We conclude: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} \cos((2k-1)\pi x)$. $\quad \triangleleft$

Examples of the Fourier Theorem (Sect. 6.2).

- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- **The Fourier Theorem: Piecewise continuous case.**
- Example: Using the Fourier Theorem.

The Fourier Theorem: Piecewise continuous case.

Recall:

Definition

A function $f : [a, b] \to \mathbb{R}$ is called *piecewise continuous* iff holds,

(a) $[a, b]$ can be partitioned in a finite number of sub-intervals such that f is continuous on the interior of these sub-intervals.

(b) f has finite limits at the endpoints of all sub-intervals.

The Fourier Theorem: Piecewise continuous case.

Theorem (Fourier Series)

If $f : [-L, L] \subset \mathbb{R} \to \mathbb{R}$ is piecewise continuous, then the function

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big]$$

where a_n and b_n given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

satisfies that:

(a) $f_F(x) = f(x)$ *for all x where f is continuous;*

(b) $f_F(x_0) = \frac{1}{2}\Big[\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x) \Big]$ *for all x_0 where f is discontinuous.*

Examples of the Fourier Theorem (Sect. 6.2).

- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- **Example: Using the Fourier Theorem.**

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Example: Using the Fourier Theorem.

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Solution: We start computing the Fourier coefficients b_n;

Example: Using the Fourier Theorem.

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Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

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$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx,$$

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$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad L = 1,$$

$$b_n = \int_{-1}^{0} (-1) \sin\big(n\pi x\big)\, dx + \int_{0}^{1} (1) \sin\big(n\pi x\big)\, dx,$$

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$$b_n = \int_{-1}^{0} (-1) \sin\big(n\pi x\big)\, dx + \int_{0}^{1} (1) \sin\big(n\pi x\big)\, dx,$$

$$b_n = \frac{(-1)}{n\pi} \Big[-\cos(n\pi x)\Big|_{-1}^{0} \Big] + \frac{1}{n\pi} \Big[-\cos(n\pi x)\Big|_{0}^{1} \Big],$$

Example: Using the Fourier Theorem.

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Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: We start computing the Fourier coefficients b_n;

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad L = 1,$$

$$b_n = \int_{-1}^{0} (-1) \sin\big(n\pi x\big)\, dx + \int_{0}^{1} (1) \sin\big(n\pi x\big)\, dx,$$

$$b_n = \frac{(-1)}{n\pi}\Big[-\cos(n\pi x)\Big|_{-1}^{0}\Big] + \frac{1}{n\pi}\Big[-\cos(n\pi x)\Big|_{0}^{1}\Big],$$

$$b_n = \frac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \frac{1}{n\pi}\big[-\cos(n\pi) + 1\big].$$

Example: Using the Fourier Theorem.

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If $n = 2k$,

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If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[1 - (-1)^{2k}\big]$,

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If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[1 - (-1)^{2k}\big]$, hence $b_{2k} = 0$.

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hence $b_{2k} = \frac{4}{(2k-1)\pi}$.

Example: Using the Fourier Theorem.

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Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, and $b_{2k} = \dfrac{4}{(2k-1)\pi}$.

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$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx,$$

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Therefore, we conclude that

$$f_F(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin\big((2k-1)\pi\, x\big).$$

$\triangleleft$