Differential linear systems (Sect. 5.4, 5.6, 5.7)

- $n \times n$ linear differential systems (5.4).
- Constant coefficients homogenoues systems (5.6).
- Examples: 2×2 linear systems (5.6).
- Phase portraits for 2×2 systems (5.7).

$n \times n$ linear differential systems (5.4).

Definition

An *$n \times n$ linear differential system* is a the following: Given an $n \times n$ matrix-valued function A, and an n-vector-valued function $\mathbf{b}$, find an n-vector-valued function $\mathbf{x}$ solution of

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t).$$

The system above is called *homogeneous* iff holds $\mathbf{b} = 0$.

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$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t) \;\Leftrightarrow\; \begin{aligned} x_1' &= a_{11}(t)\,x_1 + \cdots + a_{1n}(t)\,x_n + b_1(t) \\ &\;\vdots \\ x_n' &= a_{n1}(t)\,x_1 + \cdots + a_{nn}(t)\,x_n + b_n(t). \end{aligned}$$

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Example

Find the explicit expression for the linear system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ in the case that

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad \mathbf{b}(t) = \begin{bmatrix} e^{t} \\ 2e^{3t} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

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Solution: The 2×2 linear system is given by

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^{t} \\ 2e^{3t} \end{bmatrix}.$$

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That is,

$$x_1'(t) = x_1(t) + 3x_2(t) + e^t,$$
$$x_2'(t) = 3x_1(t) + x_2(t) + 2e^{3t}.$$

$\triangleleft$

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Compute $\mathbf{x}'$ for $\mathbf{x}(t) = \begin{bmatrix} e^{2t} \\ \sin(t) \\ \cos(t) \end{bmatrix}$.

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Remarks:

- Given an $n \times n$ matrix $A(t)$, n-vector $\mathbf{b}(t)$, find $\mathbf{x}(t)$ solution

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- We study homogeneous, constant coefficient systems, that is,
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Theorem (Diagonalizable matrix)

If $n \times n$ *matrix* A *is diagonalizable, with a linearly independent eigenvectors set* $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ *and corresponding eigenvalues* $\{\lambda_1, \cdots, \lambda_n\}$*, then the general solution* $\mathbf{x}$ *to the homogeneous, constant coefficients, linear system*

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is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$*,*

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- We solve for $\mathbf{y}(t)$ and we transform back to $\mathbf{x}(t)$.

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Find the general solution to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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Solution: Find eigenvalues and eigenvectors of A.

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Solution: We compute $\mathbf{x}^{(1)\prime}$ and then we compare it with $A\mathbf{x}^{(1)}$,

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$$\mathbf{x}^{(1)\prime}(t) = \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix}' = \begin{bmatrix} 4e^{4t} \\ 4e^{4t} \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \quad \Rightarrow \quad \mathbf{x}^{(1)\prime} = 4\mathbf{x}^{(1)}.$$

$A\mathbf{x}^{(1)}$

Examples: 2×2 linear systems (5.6).

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We conclude that $\mathbf{x}^{(1)\prime} = A\mathbf{x}^{(1)}$.

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Solution: We compute $\mathbf{x}^{(2)\prime}$

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$$A\mathbf{x}^{(2)} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-2t}$$

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So, $A\mathbf{x}^{(2)} = -2\mathbf{x}^{(2)}$. Hence, $\mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)}$. ◁

Examples: 2×2 linear systems (5.6).

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Solve the IVP $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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Therefore, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$,

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Therefore, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, hence $\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$. $\triangleleft$

Constant coefficients homogenoues systems (5.6).

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$$P^{-1}\mathbf{x}'(t) = P^{-1}A\,\mathbf{x}(t) \quad \Leftrightarrow \quad \big(P^{-1}\mathbf{x}\big)' = \big(P^{-1}AP\big)\,\big(P^{-1}\mathbf{x}\big).$$

Introduce the new unknown $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, then

$$\mathbf{y}'(t) = D\,\mathbf{y}(t) \;\Leftrightarrow\; \begin{cases} y_1'(t) = \lambda_1\,y_1(t), \\ \qquad\vdots \\ y_n'(t) = \lambda_n\,y_n(t), \end{cases}$$

Constant coefficients homogenoues systems (5.6).

Proof: Since A is diagonalizable, we know that $A = PDP^{-1}$, with

$$P = \big[\mathbf{v}_1, \cdots, \mathbf{v}_n\big], \qquad D = \mathrm{diag}\big[\lambda_1, \cdots, \lambda_n\big].$$

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Constant coefficients homogenoues systems (5.6).

Proof: Recall: $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, and $\mathbf{y}(t) = \begin{bmatrix} c_1\, e^{\lambda_1 t} \\ \vdots \\ c_n\, e^{\lambda_n t} \end{bmatrix}$.

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Transform back to $\mathbf{x}(t)$,

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Transform back to $\mathbf{x}(t)$, that is,

$$\mathbf{x}(t) = P\,\mathbf{y}(t)$$

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Transform back to $\mathbf{x}(t)$, that is,

$$\mathbf{x}(t) = P\,\mathbf{y}(t) = \big[\mathbf{v}_1, \cdots, \mathbf{v}_n\big] \begin{bmatrix} c_1\, e^{\lambda_1 t} \\ \vdots \\ c_n\, e^{\lambda_n t} \end{bmatrix}$$

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We conclude: $\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}$. $\square$

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Remark:

- $A\,\mathbf{v}_i = \lambda_i \mathbf{v}_i$.

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Remark:

- $A\,\mathbf{v}_i = \lambda_i\mathbf{v}_i$.
- The eigenvalues and eigenvectors of A are crucial to solve the differential linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$.

Differential linear systems (Sect. 5.4, 5.6, 5.7)

- $n \times n$ linear differential systems (5.4).
- Constant coefficients homogenoues systems (5.6).
- Examples: 2×2 linear systems (5.6).
- **Phase portraits for 2×2 systems (5.7).**

Phase portraits for 2×2 systems (5.7).

Remark:

- There are two main types of graphs for solutions of 2×2 linear systems:

Phase portraits for 2×2 systems (5.7).

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- Case (i): Express the solution in vector components $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, and graph x_1 and x_2 as functions of t.

Phase portraits for 2×2 systems (5.7).

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- Case (ii): Express the solution as a vector-valued function,
$$\mathbf{x}(t) = c_1\,\mathbf{v}_1\,e^{\lambda_1 t} + c_2\,\mathbf{v}_2\,e^{\lambda_2 t},$$
  and plot the vector $\mathbf{x}(t)$ for different values of t.

Phase portraits for 2×2 systems (5.7).

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- There are two main types of graphs for solutions of 2×2 linear systems:
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  and plot the vector $\mathbf{x}(t)$ for different values of t.

- Case (ii) is called a *phase portrait*.

Phase portraits for 2×2 systems (5.7).

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Phase portraits for 2×2 systems (5.7).

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Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:

We start plotting the vectors

$$\mathbf{v}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

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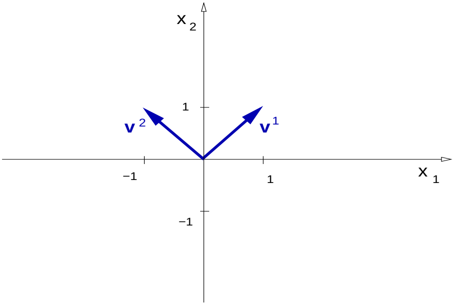

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Solution:

We now plot the functions

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t},$$

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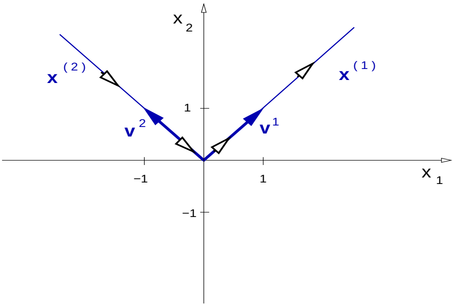

Phase portraits for 2×2 systems (5.7).

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$$-\mathbf{x}^{(1)} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t},$$

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Phase portraits for 2×2 systems (5.7).

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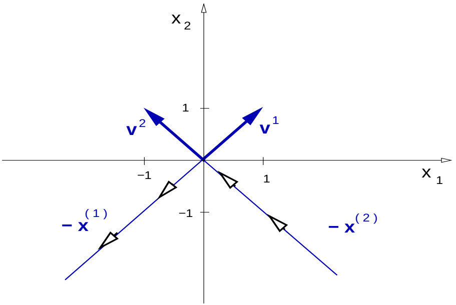

Phase portraits for 2×2 systems (5.7).

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$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:

We now plot the four functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

$$\mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)}.$$

Phase portraits for 2×2 systems (5.7).

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

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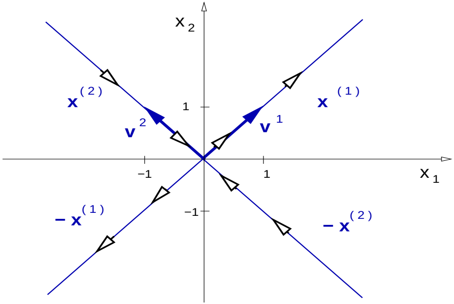

Phase portraits for 2×2 systems (5.7).

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:

We now plot the four functions

$\mathbf{x}^{(1)}, \ -\mathbf{x}^{(1)}, \ \mathbf{x}^{(2)}, \ -\mathbf{x}^{(2)},$

and $\mathbf{x}^{(1)} + \mathbf{x}^{(2)}$,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

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Plot the phase portrait of several linear combinations of the fundamental solutions found above,

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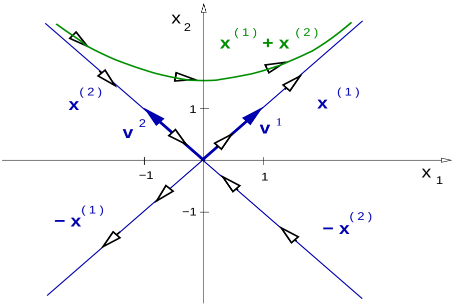

Phase portraits for 2×2 systems (5.7).

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

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Solution:

We now plot the eight functions

$$\mathbf{x}^{(1)}, \ -\mathbf{x}^{(1)}, \ \mathbf{x}^{(2)}, \ -\mathbf{x}^{(2)},$$

$$\mathbf{x}^{(1)} + \mathbf{x}^{(2)}, \quad -\mathbf{x}^{(1)} + \mathbf{x}^{(2)},$$

$$\mathbf{x}^{(1)} - \mathbf{x}^{(2)}, \quad -\mathbf{x}^{(1)} - \mathbf{x}^{(2)}.$$

Phase portraits for 2×2 systems (5.7).

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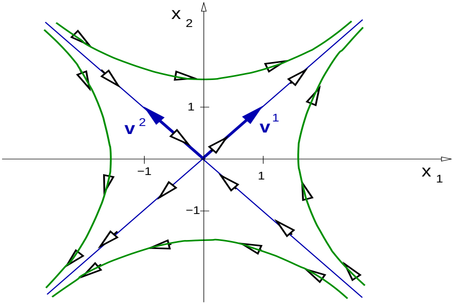

Phase portraits for 2×2 systems (5.7).

Problem:

Case (a): Consider a 2×2 matrix A having two different, real eigenvalues $\lambda_1 \neq \lambda_2$, so A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions).

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Given a solution $\mathbf{x}(t) = c_1\,\mathbf{v}_1\,e^{\lambda_1 t} + c_2\,\mathbf{v}_2\,e^{\lambda_2 t}$, to $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, plot different solution vectors $\mathbf{x}(t)$ on the plane as function of t for different choices of the constants c_1 and c_2.

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The plots are different depending on the eigenvalues signs.

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We have the following three sub-cases:

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We have the following three sub-cases:

(i) $0 < \lambda_2 < \lambda_1$, both positive;

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The plots are different depending on the eigenvalues signs.
We have the following three sub-cases:

(i) $0 < \lambda_2 < \lambda_1$, both positive;

(ii) $\lambda_2 < 0 < \lambda_1$, one positive the other negative;

Phase portraits for 2×2 systems (5.7).

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The plots are different depending on the eigenvalues signs.
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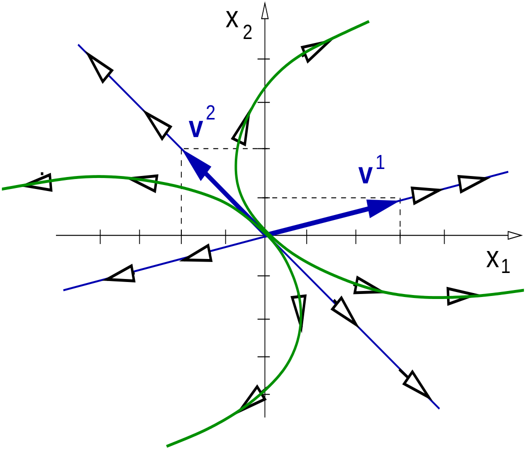
(i) $0 < \lambda_2 < \lambda_1$, both positive;

(ii) $\lambda_2 < 0 < \lambda_1$, one positive the other negative;

(iii) $\lambda_2 < \lambda_1 < 0$, both negative.

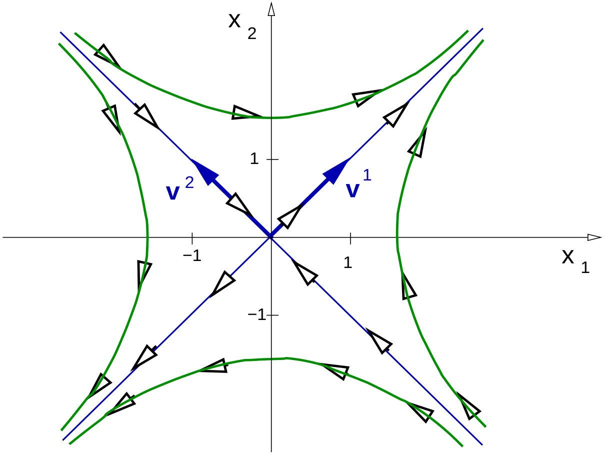
Phase portraits for 2×2 systems (5.7).

Phase portrait: Case (a), two different, real eigenvalues $\lambda_1 \neq \lambda_2$, sub-case $0 < \lambda_2 < \lambda_1$, both eigenvalue positive.



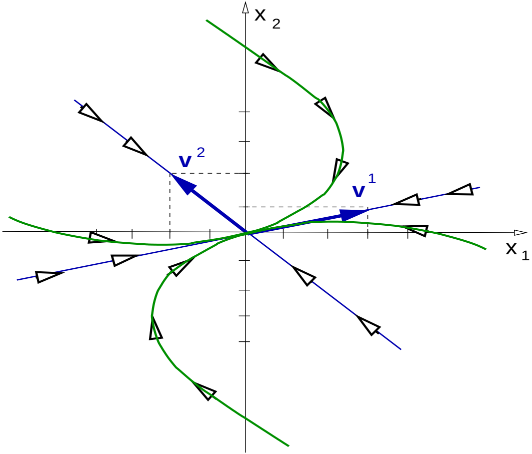
Phase portraits for 2×2 systems.

Phase portrait: Case (a), two different, real eigenvalues $\lambda_1 \neq \lambda_2$, sub-case $\lambda_2 < 0 < \lambda_1$, one eigenvalue positive the other negative.



Phase portraits for 2×2 systems (5.7).

Phase portrait: Case (a), two different, real eigenvalues $\lambda_1 \neq \lambda_2$, sub-case $\lambda_2 < \lambda_1 < 0$, both eigenvalues negative.



Complex, distinct eigenvalues (Sect. 5.8)

- Review: The case of diagonalizable matrices.
- Classification of 2×2 diagonalizable systems.
- Real matrix with a pair of complex eigenvalues.
- Phase portraits for 2×2 systems.

Review: The case of diagonalizable matrices.

Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

Review: The case of diagonalizable matrices.

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$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Review: The case of diagonalizable matrices.

Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: $\lambda_1 = 4,\ \ \mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \ \lambda_2 = -2,\ \ \mathbf{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

Review: The case of diagonalizable matrices.

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The general solution is: $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$. ◁

Complex, distinct eigenvalues (Sect. 5.8)

- Review: The case of diagonalizable matrices.
- **Classification of 2×2 diagonalizable systems.**
- Real matrix with a pair of complex eigenvalues.
- Phase portraits for 2×2 systems.

Review: Classification of 2×2 diagonalizable systems.

Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

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(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 5.9).

Remark:

(c-2) $\lambda_1 = \lambda_2$ real-valued with only one eigen-direction. Hence, A is not diagonalizable, (Section 5.9).

Complex, distinct eigenvalues (Sect. 5.8)

- Review: The case of diagonalizable matrices.
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- **Real matrix with a pair of complex eigenvalues.**
- Phase portraits for 2×2 systems.

Real matrix with a pair of complex eigenvalues.

Theorem

If $\{\lambda, \mathbf{v}\}$ *is an eigen-pair of an* $n \times n$ *real-valued matrix A, then* $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ *also is an eigen-pair of matrix A.*

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Therefore $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ is an eigen-pair of matrix A. $\square$

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Remark: The Theorem above is equivalent to the following:

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Remark: The Theorem above is equivalent to the following: If an $n \times n$ real-valued matrix A has eigen pairs

$$\lambda_1 = \alpha + i\beta, \quad \mathbf{v}_1 = \mathbf{a} + i\mathbf{b},$$

with $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then so is

$$\lambda_2 = \alpha - i\beta, \quad \mathbf{v}_2 = \mathbf{a} - i\mathbf{b}.$$

Real matrix with a pair of complex eigenvalues.

Theorem (Complex pairs)

If an $n \times n$ real-valued matrix A has eigen pairs

$$\lambda_{\pm} = \alpha \pm i\beta, \quad \mathbf{v}^{(\pm)} = \mathbf{a} \pm i\mathbf{b},$$

with $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then the differential equation

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

has a linearly independent set of two complex-valued solutions

$$\mathbf{x}^{(+)} = \mathbf{v}^{(+)}\, e^{\lambda_+ t}, \qquad \mathbf{x}^{(-)} = \mathbf{v}^{(-)}\, e^{\lambda_- t},$$

and it also has a linearly independent set of two real-valued solutions

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\, e^{\alpha t},$$

$$\mathbf{x}^{(2)} = \big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\, e^{\alpha t}.$$

Real matrix with a pair of complex eigenvalues.

Proof: We know that one solution to the differential equation is

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Euler equation implies

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Euler equation implies

$$\mathbf{x}^{(+)} = (\mathbf{a} + i\mathbf{b})\, e^{\alpha t} \left[\cos(\beta t) + i \sin(\beta t)\right],$$

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A similar calculation done on $\mathbf{x}^{(-)}$ implies

$$\mathbf{x}^{(-)} = \left[\mathbf{a} \cos(\beta t) - \mathbf{b} \sin(\beta t)\right] e^{\alpha t} - i \left[\mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t)\right] e^{\alpha t}.$$

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Introduce $\mathbf{x}^{(1)} = (\mathbf{x}^{(+)} + \mathbf{x}^{(-)})/2$,

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Introduce $\mathbf{x}^{(1)} = (\mathbf{x}^{(+)} + \mathbf{x}^{(-)})/2$, $\mathbf{x}^{(2)} = (\mathbf{x}^{(+)} - \mathbf{x}^{(-)})/(2i)$, then

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Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

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The roots of the characteristic polynomial are

$$(\lambda - 2)^2 + 9 = 0$$

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(2) Find the eigenvectors of matrix A above. For λ_+,

$$A - \lambda_+ I$$

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$$A - \lambda_+ I = A - (2 + 3i)I$$

Real matrix with a pair of complex eigenvalues.

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$$(\lambda - 2)^2 + 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} - 2 = \pm 3i \quad \Rightarrow \quad \lambda_{\pm} = 2 \pm 3i.$$

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$$A - \lambda_+ I = A - (2+3i)I = \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix}.$$

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We need to solve $(A - \lambda_{+} I)\,\mathbf{v}^{(+)} = \mathbf{0}$ for $\mathbf{v}^{(+)}$.

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Solution: $\lambda_{\pm} = 2 \pm 3i$, $(A - \lambda_+ I) = \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix}.$

We need to solve $(A - \lambda_+ I)\,\mathbf{v}^{(+)} = \mathbf{0}$ for $\mathbf{v}^{(+)}$. Gauss operations

$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

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$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \to \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

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So, the eigenvector $\mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

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$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

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So, the eigenvector $\mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is given by $v_1 = -iv_2$.

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So, the eigenvector $\mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is given by $v_1 = -iv_2$. Choose

$$v_2 = 1, \quad v_1 = -i,$$

Real matrix with a pair of complex eigenvalues.

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So, the eigenvector $\mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is given by $v_1 = -iv_2$. Choose

$$v_2 = 1, \quad v_1 = -i, \quad \Rightarrow \quad \mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \lambda_+ = 2 + 3i.$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: eigenvalues $\lambda_{\pm} = 2 \pm 3i$, and $\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Real matrix with a pair of complex eigenvalues.

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Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: eigenvalues $\lambda_{\pm} = 2 \pm 3i$, and $\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

The second eigenvector is $\mathbf{v}^{(-)} = \overline{\mathbf{v}}^{(+)}$,

Real matrix with a pair of complex eigenvalues.

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Notice that $\mathbf{v}^{(\pm)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i.$

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The notation $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$

Real matrix with a pair of complex eigenvalues.

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The notation $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b} i$ implies

$$\alpha = 2,$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: eigenvalues $\lambda_{\pm} = 2 \pm 3i$, and $\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

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Notice that $\mathbf{v}^{(\pm)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$.

The notation $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b} i$ implies

$$\alpha = 2, \qquad \beta = 3,$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: eigenvalues $\lambda_{\pm} = 2 \pm 3i$, and $\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

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Real matrix with a pair of complex eigenvalues.

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Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: $\alpha = 2$, $\beta = 3$, $\mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

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Real-valued solutions are $\mathbf{x}^{(1)} = \big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big]\, e^{\alpha t}$, and

Real matrix with a pair of complex eigenvalues.

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Real matrix with a pair of complex eigenvalues.

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$$\mathbf{x}^{(1)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \right) e^{2t}$$

Real matrix with a pair of complex eigenvalues.

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$$\mathbf{x}^{(1)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t)\right) e^{2t} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}.$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: $\alpha = 2$, $\beta = 3$, $\mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

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$$\mathbf{x}^{(1)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t)\right) e^{2t} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}.$$

$$\mathbf{x}^{(2)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(3t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(3t)\right) e^{2t}$$

Real matrix with a pair of complex eigenvalues.

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$$\mathbf{x}^{(2)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(3t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(3t) \right) e^{2t} \Rightarrow \mathbf{x}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t}.$$

$\triangleleft$

Complex, distinct eigenvalues (Sect. 5.8)

- Review: The case of diagonalizable matrices.
- Classification of 2×2 diagonalizable systems.
- Real matrix with a pair of complex eigenvalues.
- **Phase portraits for 2×2 systems.**

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Solution:

The phase portrait of the vectors

$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix},$$

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

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is a radius one circle.

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Solution:

The phase portrait of the vectors

$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix},$$

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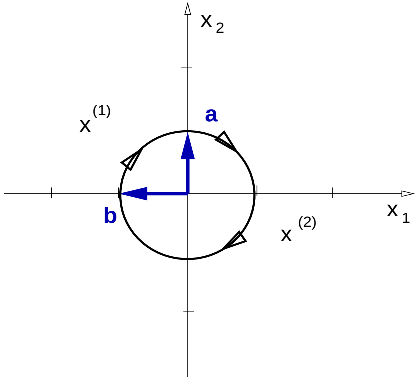

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Solution:

The phase portrait of the solutions

$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t},$$

$$\tilde{\mathbf{x}}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t},$$

are outgoing spirals.

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

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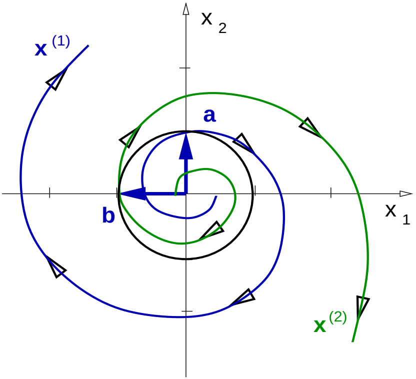

Phase portraits for 2×2 systems.

Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\,e^{\alpha t},\ \mathbf{x}^{(2)} = \big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\,e^{\alpha t}.$$

for the cases $\alpha = 0$, $\alpha > 0$, and $\alpha < 0$, where $\beta > 0$.

Phase portraits for 2×2 systems.

Example

Given any vectors $\mathbf{a}$ and $\mathbf{b}$, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big]\, e^{\alpha t}, \quad \mathbf{x}^{(2)} = \big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big]\, e^{\alpha t}.$$

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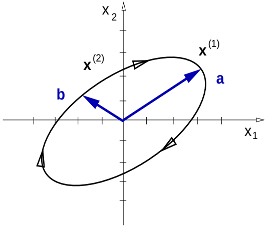


Phase portraits for 2×2 systems.

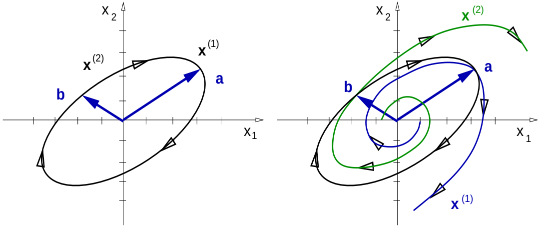
Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \left[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right] e^{\alpha t}, \ \mathbf{x}^{(2)} = \left[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right] e^{\alpha t}.$$

for the cases $\alpha = 0$, $\alpha > 0$, and $\alpha < 0$, where $\beta > 0$.

Solution:



Phase portraits for 2 × 2 systems.

Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \left[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right] e^{\alpha t}, \quad \mathbf{x}^{(2)} = \left[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right] e^{\alpha t}.$$

for the cases $\alpha = 0$, $\alpha > 0$, and $\alpha < 0$, where $\beta > 0$.

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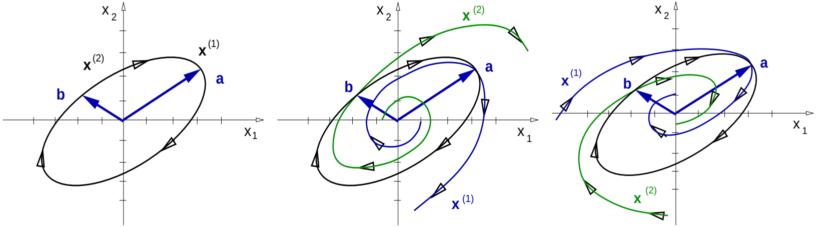

Complex, distinct eigenvalues (Sect. 5.9)

- Review: Classification of 2×2 diagonalizable systems.
- Review: The case of diagonalizable matrices.
- The algebraic multiplicity of an eigenvalue.
- Non-diagonalizable matrices with a repeated eigenvalue.
- Phase portraits for 2×2 systems.

Review: Classification of 2×2 diagonalizable systems.

Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

Review: Classification of 2×2 diagonalizable systems.

Remark:

Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

Review: Classification of 2×2 diagonalizable systems.

Remark:

Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1 = \overline{\mathbf{v}}_2$, (Section 5.8).

Review: Classification of 2×2 diagonalizable systems.

Remark:

Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1 = \overline{\mathbf{v}}_2$, (Section 5.8).

(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 5.9).

Review: Classification of 2×2 diagonalizable systems.

Remark:

Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1 = \overline{\mathbf{v}}_2$, (Section 5.8).

(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 5.9).

Remark:

(c-2) $\lambda_1 = \lambda_2$ real-valued with only one eigen-direction. Hence, A is not diagonalizable, (Section 5.9).

Complex, distinct eigenvalues (Sect. 5.9)

- Review: Classification of 2×2 diagonalizable systems.
- **Review: The case of diagonalizable matrices.**
- The algebraic multiplicity of an eigenvalue.
- Non-diagonalizable matrices with a repeated eigenvalue.
- Phase portraits for 2×2 systems.

Review: The case of diagonalizable matrices.

Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to the homogeneous, constant coefficients, linear system

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

Complex, distinct eigenvalues (Sect. 5.9)

- Review: Classification of 2×2 diagonalizable systems.
- Review: The case of diagonalizable matrices.
- **The algebraic multiplicity of an eigenvalue.**
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The algebraic multiplicity of an eigenvalue.

Definition

Let $\{\lambda_1, \cdots, \lambda_k\}$ be the set of eigenvalues of an $n \times n$ matrix, where $1 \leqslant k \leqslant n$, hence the characteristic polynomial is

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_k)^{r_k}.$$

The positive integer r_i, for $i = 1, \cdots, k$, is called the *algebraic multiplicity* of the eigenvalue λ_i. The eigenvalue λ_i is called *repeated* iff $r_i > 1$.

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Remark:

- A matrix with repeated eigenvalues may or may not be diagonalizable.

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Remark:

- A matrix with repeated eigenvalues may or may not be diagonalizable.
- Equivalently: An $n \times n$ matrix with repeated eigenvalues may or may not have a linearly independent set of n eigenvectors.

The algebraic multiplicity of an eigenvalue.

Example

Show that matrix A is diagonalizable but matrix B is not, where

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

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Solution: The eigenvalues of A are the solutions of

$$\begin{vmatrix} (3-\lambda) & 0 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix}$$

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We conclude: $\lambda_1 = 3$, $r_1 = 2$,

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Verify that the eigenvalues are: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}.$

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We conclude: A is diagonalizable.

The algebraic multiplicity of an eigenvalue.

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The algebraic multiplicity of an eigenvalue.

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The algebraic multiplicity of an eigenvalue.

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Solution: The eigenvalues of B are the solutions of

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We conclude: $\lambda_1 = 3$, $r_1 = 2$, and $\lambda_2 = 1$, $r_2 = 1$.

Verify that the eigenvalues are: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

We conclude: B is not diagonalizable.

The algebraic multiplicity of an eigenvalue.

Example

Find a fundamental set of solutions to

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

Solution: Since matrix A is diagonalizable, with eigen-pairs,

$$\lambda_1 = 3, \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad \lambda_2 = 1, \quad \left\{ \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}.$$

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We conclude that a set of fundamental solutions is

$$\left\{ \mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t},\ \mathbf{x}_2(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{3t},\ \mathbf{x}_3(t) = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} e^{t} \right\}. \quad \triangleleft$$

Complex, distinct eigenvalues (Sect. 5.9)

- Review: Classification of 2×2 diagonalizable systems.
- Review: The case of diagonalizable matrices.
- The algebraic multiplicity of an eigenvalue.
- **Non-diagonalizable matrices with a repeated eigenvalue.**
- Phase portraits for 2×2 systems.

Non-diagonalizable matrices with a repeated eigenvalue.

Theorem (Repeated eigenvalue)

If λ *is an eigenvalue of an* $n \times n$ *matrix* A *having algebraic multiplicity* $r = 2$ *and only one associated eigen-direction, then the differential equation*

$$\mathbf{x}'(t) = A\,\mathbf{x}(t),$$

has a linearly independent set of solutions given by

$$\big\{\mathbf{x}^{(1)}(t) = \mathbf{v}\ e^{\lambda t}, \quad \mathbf{x}^{(2)}(t) = \big(\mathbf{v}\ t + \mathbf{w}\big)\,e^{\lambda t}\big\}.$$

where the vector $\mathbf{w}$ *is solution of*

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$

which always has a solution $\mathbf{w}$*.*

Non-diagonalizable matrices with a repeated eigenvalue.

Recall: The case of a single second order equation

$$y'' + a_1\, y' + a_0\, y = 0$$

Non-diagonalizable matrices with a repeated eigenvalue.

Recall: The case of a single second order equation

$$y'' + a_1\, y' + a_0\, y = 0$$

with characteristic polynomial

$$p(r) = r^2 + a_1\, r + a_0$$

Non-diagonalizable matrices with a repeated eigenvalue.

Recall: The case of a single second order equation

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$$p(r) = r^2 + a_1\, r + a_0 = (r - r_1)^2.$$

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In this case a fundamental set of solutions is

$$\big\{ y_1(t) = e^{r_1 t}, \quad y_2(t) = t\, e^{r_1 t} \big\}.$$

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$$\big\{ y_1(t) = e^{r_1 t}, \quad y_2(t) = t\, e^{r_1 t} \big\}.$$

This is not the case with systems of first order linear equations,

$$\big\{ \mathbf{x}^{(1)}(t) = \mathbf{v}\; e^{\lambda t}, \quad \mathbf{x}^{(2)}(t) = \big(\mathbf{v}\; t + \mathbf{w}\big)\, e^{\lambda t} \big\}.$$

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In general, $\mathbf{w} \neq \mathbf{0}$.

Non-diagonalizable matrices with a repeated eigenvalue.

Example

Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Non-diagonalizable matrices with a repeated eigenvalue.

Example

Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Find the eigenvalues of A.

Non-diagonalizable matrices with a repeated eigenvalue.

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Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Find the eigenvalues of A. Its characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} \left(-\frac{3}{2} - \lambda\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} - \lambda\right) \end{vmatrix}$$

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So $p(\lambda) = \lambda^2 + 2\lambda + 1$

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So $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$. The roots and multiplicity are

$$\lambda = -1, \qquad r = 2.$$

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The corresponding eigenvectors are the solutions of $(A + I)\mathbf{v} = \mathbf{0}$,

Non-diagonalizable matrices with a repeated eigenvalue.

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$$\begin{bmatrix} \left(-\frac{3}{2} + 1\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} + 1\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

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Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \dfrac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Recall: $\lambda = -1$, with $r = 2$, and $(A + I) \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

Non-diagonalizable matrices with a repeated eigenvalue.

Example

Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Recall: $\lambda = -1$, with $r = 2$, and $(A + I) \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

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$$\lambda = -1, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_2.$$

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We conclude that this eigenvalue has only one eigen-direction.

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$$\left[\begin{array}{cc|c} -\frac{1}{2} & 1 & 2 \\ -\frac{1}{4} & \frac{1}{2} & 1 \end{array}\right]$$

Non-diagonalizable matrices with a repeated eigenvalue.

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Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \dfrac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

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We obtain $w_1 = 2w_2 - 4$.

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We obtain $w_1 = 2w_2 - 4$. That is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

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Given a solution $\mathbf{w}$, then $c\mathbf{v} + \mathbf{w}$ is also a solution, $c \in \mathbb{R}$.

Non-diagonalizable matrices with a repeated eigenvalue.

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We choose the simplest solution, $\mathbf{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

Non-diagonalizable matrices with a repeated eigenvalue.

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We obtain $w_1 = 2w_2 - 4$. That is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

Given a solution $\mathbf{w}$, then $c\mathbf{v} + \mathbf{w}$ is also a solution, $c \in \mathbb{R}$.

We choose the simplest solution, $\mathbf{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$. We conclude,

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}, \qquad \mathbf{x}^{(2)}(t) = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}. \qquad \triangleleft$$

Non-diagonalizable matrices with a repeated eigenvalue.

Example

Find the solution $\mathbf{x}$ to the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Non-diagonalizable matrices with a repeated eigenvalue.

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$$\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \frac{1}{4} \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution: The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$$

Non-diagonalizable matrices with a repeated eigenvalue.

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Find the solution $\mathbf{x}$ to the IVP

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The initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Non-diagonalizable matrices with a repeated eigenvalue.

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Non-diagonalizable matrices with a repeated eigenvalue.

Example

Find the solution $\mathbf{x}$ to the IVP

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution: The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$$

The initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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Find the solution $\mathbf{x}$ to the IVP

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \frac{1}{4} \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

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$$\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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Find the solution $\mathbf{x}$ to the IVP

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We conclude: $\mathbf{x}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{4} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$ $\triangleleft$

Complex, distinct eigenvalues (Sect. 5.9)

- Review: Classification of 2×2 diagonalizable systems.
- Review: The case of diagonalizable matrices.
- The algebraic multiplicity of an eigenvalue.
- Non-diagonalizable matrices with a repeated eigenvalue.
- **Phase portraits for 2×2 systems.**

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of

$$\mathbf{x}' = A\mathbf{x}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

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We start plotting the vectors

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

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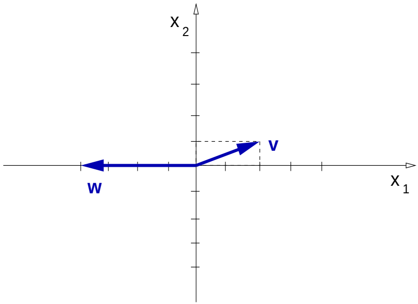

Phase portraits for 2×2 systems.

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Now plot the solutions

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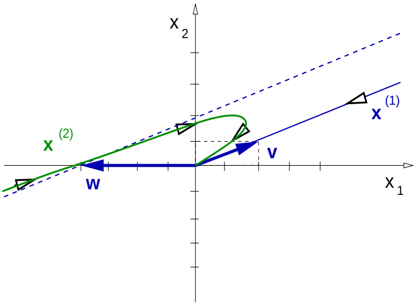

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$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

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This is the case $\lambda < 0$.

Phase portraits for 2×2 systems.

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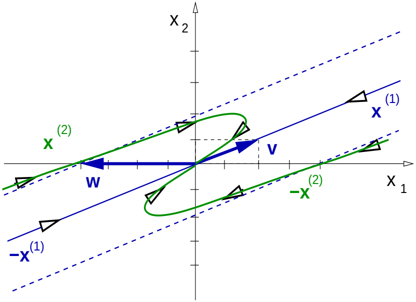

Phase portraits for 2×2 systems.

Example

Given any vectors $\mathbf{v}$ and $\mathbf{w}$, and any constant λ, plot the phase portraits of the functions

$$\mathbf{x}^{(1)}(t) = \mathbf{v}\, e^{\lambda t}, \qquad \mathbf{x}^{(2)}(t) = \big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{\lambda t},$$

Solution:

The case $\lambda < 0$. We plot the functions

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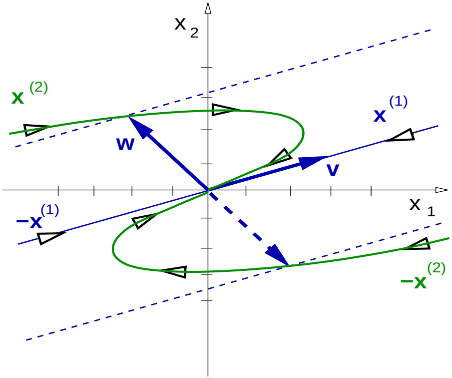

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