# The Laplace Transform of step functions (Sect. 4.3).

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.

# Overview and notation.

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A function $u$ is called a *step function* at $t = 0$ iff holds

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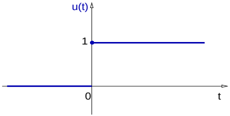

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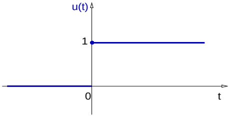


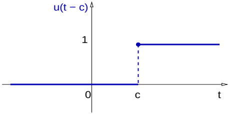

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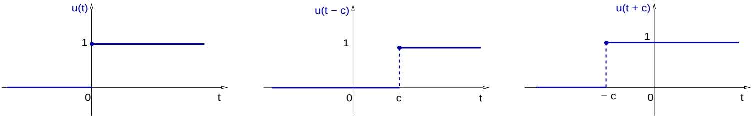


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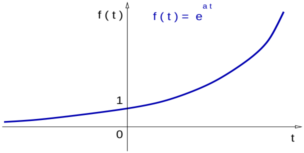

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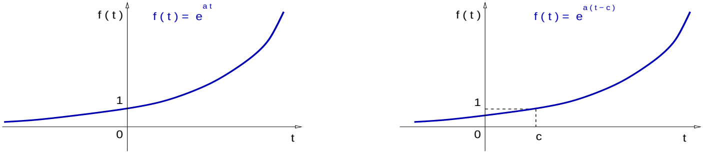

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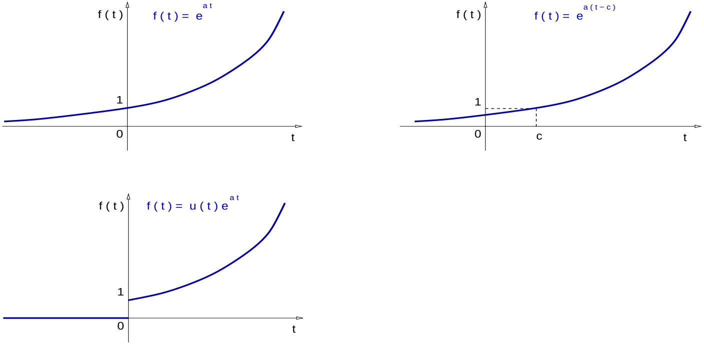

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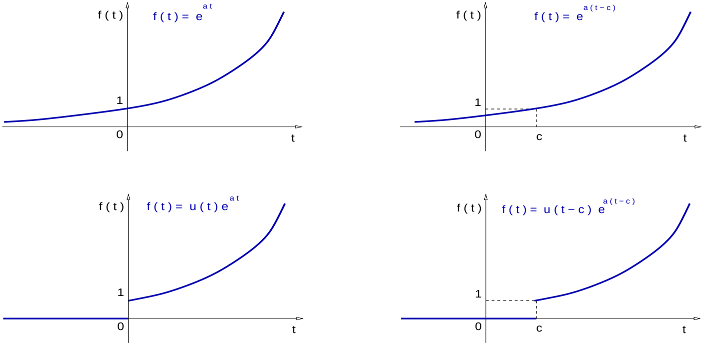

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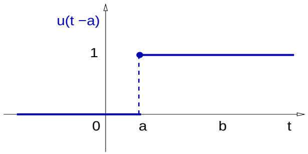

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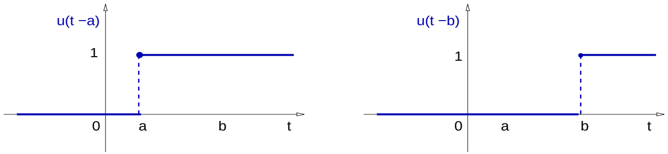

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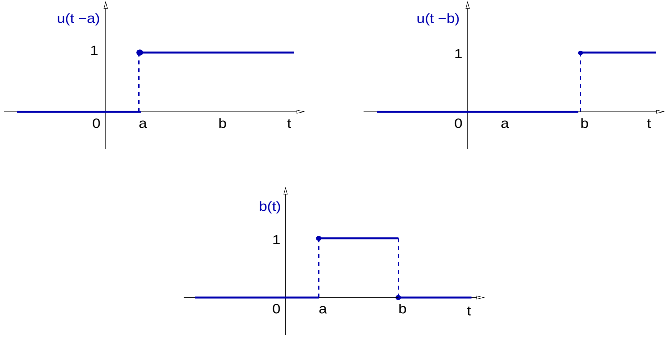


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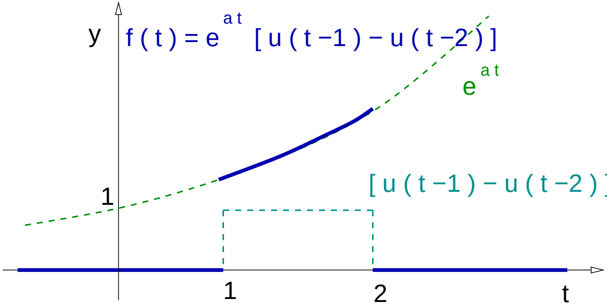

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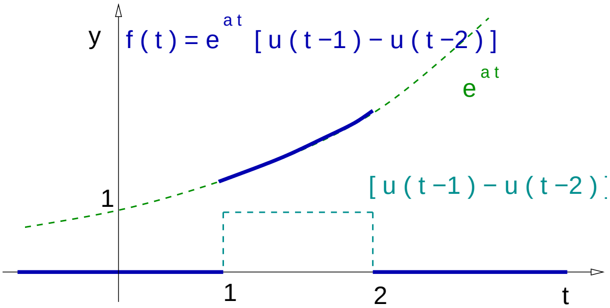


Notation: It is common in the literature to denote the function values $u(t-c)$ as $u_c(t)$.

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Theorem

*Given any real number $c \geqslant 0$, the following equation holds,*

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We conclude that $\mathcal{L}[u(t-c)] = \dfrac{e^{-cs}}{s}$. $\square$

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*If* $F(s) = \mathcal{L}[f(t)]$ *exists for* $s > a \geqslant 0$ *and* $c \geqslant 0$*, then holds*

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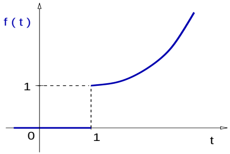

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We conclude: $\mathcal{L}[f(t)] = \dfrac{e^{-s}}{s^3}\big(2 + s^2\big)$. $\triangleleft$

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# The Laplace Transform of step functions (Sect. 4.3).

## Last Lecture

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.

## This Lecture

- Differential equations with discontinuous sources.

# Equations with discontinuous sources (Sect. 4.3).

- Differential equations with discontinuous sources.
- We solve the IVPs:
  (a) Example 1:
$$y' + 2y = u(t-4), \qquad y(0) = 3.$$
  (b) Example 2:
$$y'' + y' + \frac{5}{4}\,y = b(t), \quad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$
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# Equations with discontinuous sources (Sect. 4.3).

- Differential equations with discontinuous sources.
- We solve the IVPs:

(a) **Example 1:**

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

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We get, $a + b = 0$, $2a = 1$. We obtain: $a = \dfrac{1}{2}$, $b = -\dfrac{1}{2}$. Hence,

$$\frac{1}{s(s+2)} = \frac{1}{2}\left[\frac{1}{s} - \frac{1}{(s+2)}\right].$$

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The algebraic equation for $\mathcal{L}[y]$ has the form,

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We conclude that

$$y(t) = 3e^{-2t} + \frac{1}{2}\,u(t-4)\Big[1 - e^{-2(t-4)}\Big]. \qquad \triangleleft$$

# Equations with discontinuous sources (Sect. 4.3).

- Differential equations with discontinuous sources.
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Rewrite the source function using
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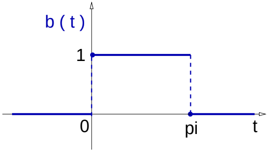

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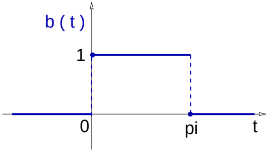


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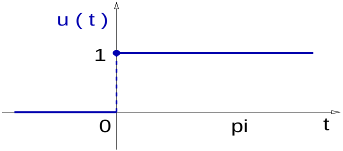

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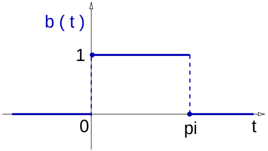


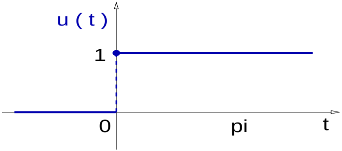


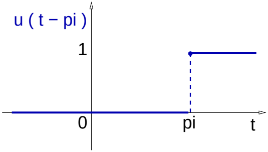

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Solution: The graphs imply: $b(t) = u(t) - u(t - \pi)$

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$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \frac{1}{s}.$$

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Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}.$

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# Differential equations with discontinuous sources.

## Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}.$

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$ and $\mathcal{L}[y'] = s\,\mathcal{L}[y]$.

Therefore, $\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}.$

# Differential equations with discontinuous sources.

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We arrive at the expression: $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$.

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we obtain, $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) H(s)$.

In other words: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\, H(s)\big]$.

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We only need to find $h(t) = \mathcal{L}^{-1}\Big[\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big]$.

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This equation implies that $a$, $b$, and $c$, are solutions of

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We have to compute the inverse Laplace Transform

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In this case we complete the square in the denominator,

# Differential equations with discontinuous sources.

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So: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s} - \frac{(s+1)}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big].$

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In this case we complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)\,s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4} = \Big(s + \frac{1}{2}\Big)^2 + 1.$$

So: $h(t) = \dfrac{4}{5}\,\mathcal{L}^{-1}\Big[\dfrac{1}{s} - \dfrac{(s+1)}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big].$

That is, $h(t) = \dfrac{4}{5}\,\mathcal{L}^{-1}\Big[\dfrac{1}{s}\Big] - \dfrac{4}{5}\,\mathcal{L}^{-1}\Big[\dfrac{\big(s + \frac{1}{2}\big) + \frac{1}{2}}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big].$

# Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\, y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\, \mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\, \mathcal{L}^{-1}\Big[\frac{\big(s + \frac{1}{2}\big) + \frac{1}{2}}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big].$

# Differential equations with discontinuous sources.

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$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\big(s+\frac{1}{2}\big)}{\big[\big(s+\frac{1}{2}\big)^2+1\big]}\Big] - \frac{2}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{\big[\big(s+\frac{1}{2}\big)^2+1\big]}\Big].$$

# Differential equations with discontinuous sources.

## Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\big(s+\frac{1}{2}\big)}{\big[\big(s+\frac{1}{2}\big)^2 + 1\big]}\Big] - \frac{2}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{\big[\big(s+\frac{1}{2}\big)^2 + 1\big]}\Big].$$

Recall: $\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t).$

# Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\big(s+\frac{1}{2}\big)+\frac{1}{2}}{\big[\big(s+\frac{1}{2}\big)^2+1\big]}\Big].$

$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\big(s+\frac{1}{2}\big)}{\big[\big(s+\frac{1}{2}\big)^2+1\big]}\Big] - \frac{2}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{\big[\big(s+\frac{1}{2}\big)^2+1\big]}\Big].$$

Recall: $\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t)$. Hence,

$$h(t) = \frac{4}{5}\,\Big[1 - e^{-t/2}\,\cos(t) - \frac{1}{2}\,e^{-t/2}\,\sin(t)\Big].$$

# Differential equations with discontinuous sources.

## Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\left(s + \frac{1}{2}\right) + \frac{1}{2}}{\left[\left(s + \frac{1}{2}\right)^2 + 1\right]}\Big]$.

$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\left(s + \frac{1}{2}\right)}{\left[\left(s + \frac{1}{2}\right)^2 + 1\right]}\Big] - \frac{2}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{\left[\left(s + \frac{1}{2}\right)^2 + 1\right]}\Big].$$

Recall: $\mathcal{L}^{-1}\big[F(s - c)\big] = e^{ct}\,f(t)$. Hence,

$$h(t) = \frac{4}{5}\,\Big[1 - e^{-t/2}\,\cos(t) - \frac{1}{2}\,e^{-t/2}\,\sin(t)\Big].$$

We conclude: $y(t) = h(t) + u(t - \pi)h(t - \pi)$. $\triangleleft$

# Equations with discontinuous sources (Sect. 4.3).

- Differential equations with discontinuous sources.
- We solve the IVPs:
  (a) Example 1:

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

  (b) Example 2:

$$y'' + y' + \frac{5}{4}\,y = b(t), \quad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad b(t) = \begin{cases} 1, & t \in [0,\pi) \\ 0, & t \in [\pi,\infty). \end{cases}$$

  (c) **Example 3:**

$$y'' + y' + \frac{5}{4}\,y = g(t), \quad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t), & t \in [0,\pi) \\ 0, & t \in [\pi,\infty). \end{cases}$$

# Differential equations with discontinuous sources.

## Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

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Solution:

Rewrite the source function using step functions.

# Differential equations with discontinuous sources.

## Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

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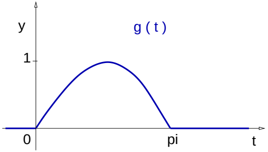

# Differential equations with discontinuous sources.

## Example

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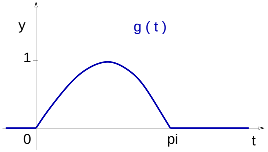


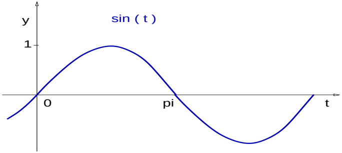

# Differential equations with discontinuous sources.

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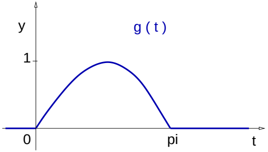


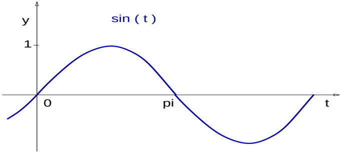


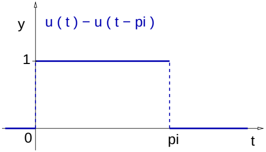

# Differential equations with discontinuous sources.

## Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $g(t) = \big[u(t) - u(t-\pi)\big]\,\sin(t).$

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$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $g(t) = \big[u(t) - u(t-\pi)\big]\,\sin(t)$.

Recall the identity: $\sin(t) = -\sin(t-\pi)$.

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Use the Laplace transform to find the solution of the IVP

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Solution: The graphs imply: $g(t) = \big[u(t) - u(t-\pi)\big]\,\sin(t)$.

Recall the identity: $\sin(t) = -\sin(t-\pi)$. Then,

$$g(t) = u(t)\,\sin(t) - u(t-\pi)\,\sin(t),$$

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Recall the identity: $\sin(t) = -\sin(t-\pi)$. Then,

$$g(t) = u(t)\,\sin(t) - u(t-\pi)\,\sin(t),$$

$$g(t) = u(t)\,\sin(t) + u(t-\pi)\,\sin(t-\pi).$$

# Differential equations with discontinuous sources.

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Recall the identity: $\sin(t) = -\sin(t-\pi)$. Then,

$$g(t) = u(t)\,\sin(t) - u(t-\pi)\,\sin(t),$$

$$g(t) = u(t)\,\sin(t) + u(t-\pi)\,\sin(t-\pi).$$

Now is simple to find $\mathcal{L}[g]$,

# Differential equations with discontinuous sources.

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Solution: The graphs imply: $g(t) = \big[u(t) - u(t - \pi)\big]\,\sin(t)$.

Recall the identity: $\sin(t) = -\sin(t - \pi)$. Then,

$$g(t) = u(t)\,\sin(t) - u(t - \pi)\,\sin(t),$$

$$g(t) = u(t)\,\sin(t) + u(t - \pi)\,\sin(t - \pi).$$

Now is simple to find $\mathcal{L}[g]$, since

$$\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t - \pi)\,\sin(t - \pi)].$$

# Differential equations with discontinuous sources.

## Example

Use the Laplace transform to find the solution of the IVP

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Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

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Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

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Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

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Recall the Laplace transform of the differential equation

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

# Differential equations with discontinuous sources.

## Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0,\pi) \\ 0 & t \in [\pi,\infty). \end{cases}$$

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Recall the Laplace transform of the differential equation

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The initial conditions imply:

# Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

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Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

Recall the Laplace transform of the differential equation

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$

# Differential equations with discontinuous sources.

## Example

Use the Laplace transform to find the solution of the IVP

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Recall the Laplace transform of the differential equation

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$ and $\mathcal{L}[y'] = s\,\mathcal{L}[y]$.

# Differential equations with discontinuous sources.

## Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

Recall the Laplace transform of the differential equation

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$ and $\mathcal{L}[y'] = s\,\mathcal{L}[y]$.

Therefore, $\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \left(1 + e^{-\pi s}\right)\frac{1}{(s^2+1)}$.

# Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

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$$\mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \frac{1}{\left(s^2 + s + \frac{5}{4}\right)\left(s^2 + 1\right)}.$$

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$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\left(s^2 + s + \frac{5}{4}\right) \mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \frac{1}{(s^2+1)}$.

$$\mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2+1)}.$$

Introduce the function $H(s) = \dfrac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2+1)}.$

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$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\left(s^2 + s + \frac{5}{4}\right) \mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \frac{1}{(s^2+1)}$.

$$\mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2+1)}.$$

Introduce the function $H(s) = \dfrac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2+1)}$.

Then, $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)]$.

# Differential equations with discontinuous sources.

## Example

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$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)]$, and

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)\left(s^2 + 1\right)}.$$

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## Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\, y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\, H(s)]$, and

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)\left(s^2 + 1\right)}.$$

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Therefore, we get

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This equation implies that $a$, $b$, $c$, and $d$, are solutions of

$$a + c = 0, \quad b + c + d = 0, \quad a + \frac{5}{4}\,c + d = 0, \quad b + \frac{5}{4}\,d = 1.$$

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Solution: So: $a = \dfrac{16}{17}$, $b = \dfrac{12}{17}$, $c = -\dfrac{16}{17}$, $d = \dfrac{4}{17}$.

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Complete the square in the denominator,

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We conclude: $y(t) = h(t) + u(t-\pi)h(t-\pi)$. $\triangleleft$

# Generalized sources (Sect. 4.4).

- The Dirac delta generalized function.
- Properties of Dirac's delta.
- Relation between deltas and steps.
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Consider the sequence of functions for $n \geqslant 1$,

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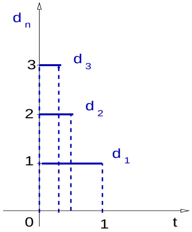

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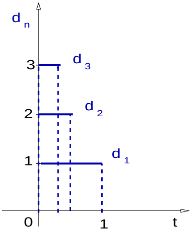


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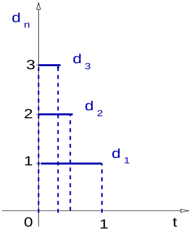


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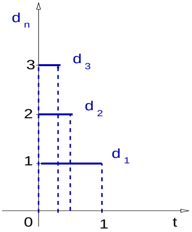


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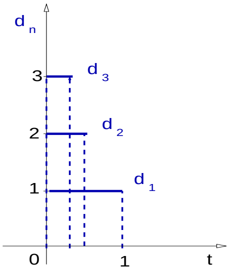

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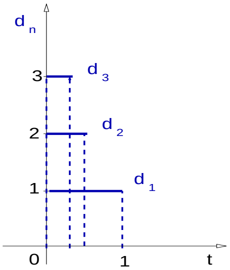


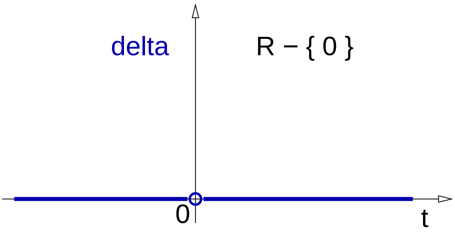

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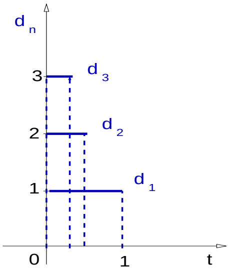


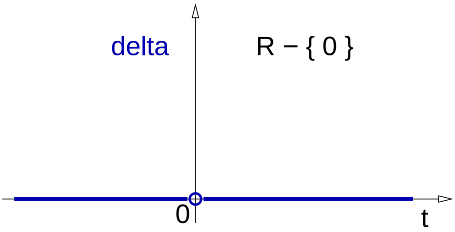


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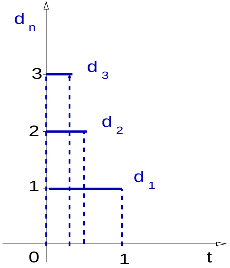


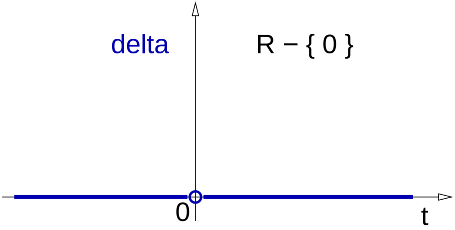


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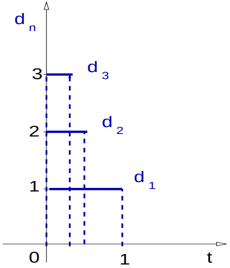


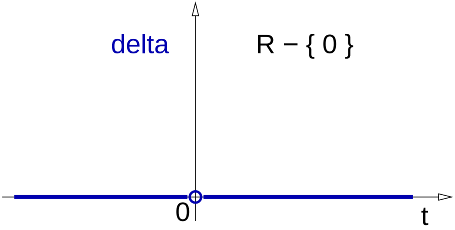


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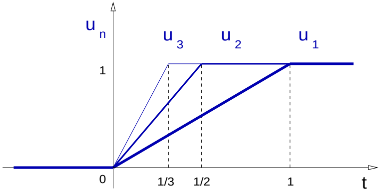

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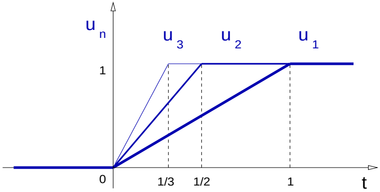


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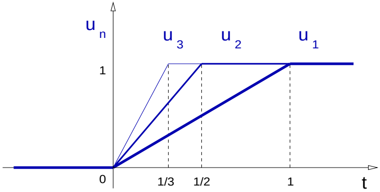


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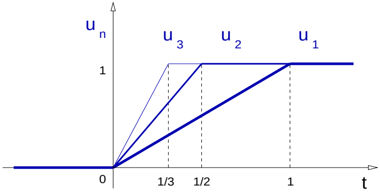


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- Properties of Dirac's delta.
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that is, $\mathcal{L}[y] = \dfrac{e^{-\pi s}}{2}\,\dfrac{2}{(s^2 + 4)} - \dfrac{e^{-2\pi s}}{2}\,\dfrac{2}{(s^2 + 4)}$.

# Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

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Recall: $e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u(t-c)\,f(t-c)]$.

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Recall: $e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u(t-c)\,f(t-c)]$. Therefore,

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We conclude: $y(t) = \frac{1}{2}\big[u(t-\pi) - u(t-2\pi)\big]\,\sin(2t).$ $\triangleleft$