Review for Exam 2.

- 6 or 7 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Variation of parameters (2.6).
  - Undetermined coefficients (2.5).
  - Constant coefficients, homogeneous, (2.2)-(2.4).
  - Reduction order method, (2.4.2).
  - Second order variable coefficients, (2.1).
  - First order homogeneous (1.3.2).

Review for Exam 2.

Notation for webwork: Consider the equation:

$$y'' + a_1\, y' + a_2\, y = 0.$$

Let r_+, r_- be the roots of the characteristic polynomial.

- If $r_+ > r_-$ real, then
  - First fundamental solution: $y_1(t) = e^{r_+ t}$.
  - Second fundamental solution: $y_2(t) = e^{r_- t}$.
- If $r_\pm = \alpha \pm i\beta$ complex, then
  - First fundamental solution: $y_1(t) = e^{\alpha t} \cos(\beta t)$.
  - Second fundamental solution: $y_2(t) = e^{\alpha t} \sin(\beta t)$.
- If $r_+ = r_- = r$ real, then
  - First fundamental solution: $y_1(t) = e^{rt}$.
  - Second fundamental solution: $y_2(t) = t\, e^{rt}$.

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  - **Variation of parameters (2.6).**
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  - Second order variable coefficients, (2.1).
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Variation of parameters (2.6).

Example

Find a particular solution of the equation

$$x^2\, y'' - 6x\, y' + 10\, y = 2x^{10},$$

knowing that $y_1 = x^5$ and $y_2 = x^2$ are solutions to the homogeneous equation.

Variation of parameters (2.6).

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knowing that $y_1 = x^5$ and $y_2 = x^2$ are solutions to the homogeneous equation.

Solution: We first need to divide the equation by x^2,

$$y'' - \frac{6}{x}\, y' + \frac{10}{x^2}\, y = 2x^8,$$

Variation of parameters (2.6).

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Solution: We first need to divide the equation by x^2,

$$y'' - \frac{6}{x}\, y' + \frac{10}{x^2}\, y = 2x^8,$$

Then the source function is $f(x) = 2x^8$.

Variation of parameters (2.6).

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$$y'' - \frac{6}{x}\, y' + \frac{10}{x^2}\, y = 2x^8,$$

Then the source function is $f(x) = 2x^8$. We now compute the Wronskian of y_1, y_2,,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

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$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^5 & x^2 \\ 5x^4 & 2x \end{vmatrix}$$

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Hence $W = -3x^6$.

Variation of parameters (2.6).

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Solution: $y_1 = x^5, \;\; y_2 = x^2, \;\; f = 2x^8, \;\; W = -3x^6.$

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Now we find the functions u_1 and u_2,

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$$u_1' = -\frac{y_2 f}{W}$$

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Find a particular solution of the equation

$$x^2 y'' - 6x y' + 10 y = 2x^{10},$$

knowing that $y_1 = x^5$ and $y_2 = x^2$ are solutions to the homogeneous equation.

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Now we find the functions u_1 and u_2,

$$u_1' = -\frac{y_2 f}{W} = -\frac{x^2\,2x^8}{(-3)x^6} = \frac{2}{3}\,x^4$$

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$$u_1' = -\frac{y_2 f}{W} = -\frac{x^2\, 2x^8}{(-3)x^6} = \frac{2}{3}\, x^4 \quad \Rightarrow \quad u_1 = \frac{2}{15}\, x^5.$$

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$$u_2' = \frac{y_1 f}{W} = \frac{x^5 2x^8}{(-3)x^6} = -\frac{2}{3} x^7$$

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$$x^2\, y'' - 6x\, y' + 10\, y = 2x^{10},$$

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$$u_2' = \frac{y_1 f}{W} = \frac{x^5 2x^8}{(-3)x^6} = -\frac{2}{3} x^7 \quad \Rightarrow \quad u_2 = -\frac{2}{24}\, x^8.$$

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$$y_p = u_1 y_1 + u_2 y_2$$

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Now we find the functions u_1 and u_2,

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$$y_p = u_1 y_1 + u_2 y_2 = \frac{2}{15}\, x^5\, x^5 - \frac{2}{24}\, x^8\, x^2$$

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$$y_p = u_1 y_1 + u_2 y_2 = \frac{2}{15}\,x^5\,x^5 - \frac{2}{24}\,x^8\,x^2 = \frac{2}{3}\,x^{10}\Big(\frac{1}{5} - \frac{1}{8}\Big)$$

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that is, $y_p = \dfrac{2}{3}\, x^{10} \Big(\dfrac{8-5}{40}\Big),$

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Example

Find a particular solution of the equation

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knowing that $y_1 = x^5$ and $y_2 = x^2$ are solutions to the homogeneous equation.

Solution: $y_1 = x^5$, $y_2 = x^2$, $f = 2x^8$, $W = -3x^6$.
Now we find the functions u_1 and u_2,

$$u_1' = -\frac{y_2 f}{W} = -\frac{x^2\,2x^8}{(-3)x^6} = \frac{2}{3}\,x^4 \quad \Rightarrow \quad u_1 = \frac{2}{15}\,x^5.$$

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$$y_p = u_1 y_1 + u_2 y_2 = \frac{2}{15}\,x^5\,x^5 - \frac{2}{24}\,x^8\,x^2 = \frac{2}{3}\,x^{10}\Big(\frac{1}{5} - \frac{1}{8}\Big)$$

that is, $y_p = \frac{2}{3}\,x^{10}\Big(\frac{8-5}{40}\Big)$, hence, $y_p = \frac{1}{20}\,x^{10}$. ◁

Variation of parameters (2.6).

Example

Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2}\, e^{-2x}.$$

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$$r^2 + 4r + 4 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2}\left[-4 \pm \sqrt{16 - 16}\right]$$

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Fundamental solutions of the homogeneous equations are

$$y_1 = e^{-2x}, \quad y_2 = x\, e^{-2x}.$$

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We now compute their Wronskian,

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Hence $W = e^{-4x}$.

Variation of parameters (2.6).

Example

Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2}\, e^{-2x}.$$

Solution: $y_1 = e^{-2x}$, $y_2 = x\, e^{-2x}$, $g = x^{-2}\, e^{-2x}$, $W = e^{-4x}$.

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Now we find the functions u_1 and u_2,

$$u_1' = -\frac{y_2 g}{W} = -\frac{x\, e^{-2x}\, x^{-2}\, e{-2x}}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln|x|.$$

$$u_2' = \frac{y_1 g}{W} = \frac{e^{-2x}\, x^{-2}\, e{-2x}}{e^{-4x}}$$

Variation of parameters (2.6).

Example

Use the variation of parameters to find the general solution of

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$y_p = u_1 y_1 + u_2 y_2$

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$$y_p = u_1 y_1 + u_2 y_2 = -\ln|x|\, e^{-2x} - \frac{1}{x}\, x e^{-2x} = -(1 + \ln|x|)\, e^{-2x}.$$

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Use the variation of parameters to find the general solution of

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Since $\tilde{y}_p = -\ln|x|\, e^{-2x}$ is solution,

Variation of parameters (2.6).

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$$y_p = u_1 y_1 + u_2 y_2 = -\ln|x|\, e^{-2x} - \frac{1}{x}\, x e^{-2x} = -(1 + \ln|x|)\, e^{-2x}.$$

Since $\tilde{y}_p = -\ln|x|\, e^{-2x}$ is solution, $y = (c_1 + c_2 x - \ln|x|)\, e^{-2x}$. $\lhd$

Review for Exam 2.

- 5 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Variation of parameters (2.6).
  - **Undetermined coefficients (2.5).**
  - Constant coefficients, homogeneous, (2.2)-(2.4).
  - Reduction order method, (2.4.2).
  - Second order variable coefficients, (2.1).
  - First order homogeneous (1.3.2).

Undetermined coefficients (2.5).

Guessing Solution Table.

$f_i(t)$ $(K,\ m,\ a,\ b,$ given.)	$y_{p_i}(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
Kt^m	$k_m t^m + k_{m-1} t^{m-1} + \cdots + k_0$
$K\cos(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$K\sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$Kt^m e^{at}$	$e^{at}(k_m t^m + \cdots + k_0)$
$Ke^{at}\cos(bt)$	$e^{at}\big[k_1 \cos(bt) + k_2 \sin(bt)\big]$
$KKe^{at}\sin(bt)$	$e^{at}\big[k_1 \cos(bt) + k_2 \sin(bt)\big]$
$Kt^m\cos(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1 \cos(bt) + a_2 \sin(bt)\big]$
$Kt^m\sin(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1 \cos(bt) + a_2 \sin(bt)\big]$

Undetermined coefficients (2.5).

Example

Find a particular solution to

$$y'' + 2y' - 2y = e^{-4it}.$$

Using this solution find particular solutions to the equations

$$y'' + 2y' - 2y = \cos(-4t), \qquad y'' + 2y' - 2y = \sin(-4t).$$

Undetermined coefficients (2.5).

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Solution: Since the source is and exponential $f(t) = e^{-4it}$,

Undetermined coefficients (2.5).

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Solution: Since the source is and exponential $f(t) = e^{-4it}$, we guess as particular solution the exponential $y_p(t) = k\, e^{-4it}$.

Undetermined coefficients (2.5).

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Solution: Since the source is and exponential $f(t) = e^{-4it}$, we guess as particular solution the exponential $y_p(t) = k\, e^{-4it}$. We now check whether y_p is solution ot the homogeneous eq.:

Undetermined coefficients (2.5).

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Solution: Since the source is and exponential $f(t) = e^{-4it}$, we guess as particular solution the exponential $y_p(t) = k\, e^{-4it}$. We now check whether y_p is solution ot the homogeneous eq.:

$$r^2 + 2r - 2 = 0$$

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$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2}\left[-2 \pm \sqrt{4+8}\right]$$

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$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big[-2 \pm \sqrt{4+8}\big] \quad \Rightarrow \quad \text{Real roots.}$$

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$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2}\big[-2 \pm \sqrt{4+8}\big] \quad \Rightarrow \quad \text{Real roots.}$$

Hence y_p is not solution of the homogeneous equation.

Undetermined coefficients (2.5).

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Solution: Recall: $y_p(t) = k\, e^{-4it}$.

Undetermined coefficients (2.5).

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Solution: Recall: $y_p(t) = k\, e^{-4it}$.

$$\big[(-4i)^2 + 2(-4i) - 2\big]\, ke^{-4it} = e^{-4it}$$

Undetermined coefficients (2.5).

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Solution: Recall: $y_p(t) = k\, e^{-4it}$.

$$\big[(-4i)^2 + 2(-4i) - 2\big]\, k e^{-4it} = e^{-4it} \quad \Rightarrow \quad (-16 - 8i - 2)k = 1$$

Undetermined coefficients (2.5).

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$$k = -\frac{1}{18 + 8i}$$

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$$\big[(-4i)^2 + 2(-4i) - 2\big]\, k e^{-4it} = e^{-4it} \quad \Rightarrow \quad (-16 - 8i - 2)k = 1$$

$$k = -\frac{1}{18 + 8i} = -\frac{1}{2}\,\frac{1}{(9 + 4i)}\,\frac{(9 - 4i)}{(9 - 4i)} = -\frac{1}{2}\,\frac{(9 - 4i)}{(9^2 + 4^2)}.$$

Hence, $y_p(t) = -\dfrac{1}{2(9^2 + 4^2)}\,(9 - 4i)\, e^{-4it}$.

Undetermined coefficients (2.5).

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Find a particular solution to

$$y'' + 2y' - 2y = e^{-4it}.$$

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For the second part of the problem,

Undetermined coefficients (2.5).

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For the second part of the problem, we need to compute the real and imaginary parts of or solution:

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$$y_p(t) = -\frac{1}{2(9^2 + 4^2)}(9 - 4i)\big[\cos(4t) - i\sin(4t)\big]$$

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$$y_{p_r} = -\frac{1}{2(9^2 + 4^2)}\,\big[9\cos(4t) - 4\sin(4t)\big]$$

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$$y_{p_r} = -\frac{1}{2(9^2 + 4^2)}\,\big[9\cos(4t) - 4\sin(4t)\big]$$

$$y_{p_i} = -\frac{1}{2(9^2 + 4^2)}\,\big[-4\cos(4t) - 9\sin(4t)\big]$$

Undetermined coefficients (2.5).

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Undetermined coefficients (2.5).

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Solution: We know that the general solution to homogeneous equation is $y(t) = c_1 e^{4t} + c_2 e^{-t}$.

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Following the table: Since $f = 2\sin(t)$,

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This guess satisfies $L(y_p) \neq 0$.

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This guess satisfies $L(y_p) \neq 0$.

Compute: $y_p' = k_1 \cos(t) - k_2 \sin(t)$, $y_p'' = -k_1 \sin(t) - k_2 \cos(t)$.

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Compute: $y_p' = k_1 \cos(t) - k_2 \sin(t)$, $y_p'' = -k_1 \sin(t) - k_2 \cos(t)$.

$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$

$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

Undetermined coefficients (2.5).

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$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

$$(-5k_1 + 3k_2)\sin(t) + (-3k_1 - 5k_2)\cos(t) = 2\sin(t).$$

Undetermined coefficients (2.5).

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: Recall:

$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$
$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

$$(-5k_1 + 3k_2)\sin(t) + (-3k_1 - 5k_2)\cos(t) = 2\sin(t).$$

This equation holds for all $t \in \mathbb{R}$. In particular, at $t = \frac{\pi}{2}$, $t = 0$.

$$-5k_1 + 3k_2 = 2,$$
$$-3k_1 - 5k_2 = 0,$$

Undetermined coefficients (2.5).

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: Recall:

$$L(y_p) = [-k_1\sin(t) - k_2\cos(t)] - 3[k_1\cos(t) - k_2\sin(t)]$$
$$-4[k_1\sin(t) + k_2\cos(t)] = 2\sin(t),$$

$$(-5k_1 + 3k_2)\sin(t) + (-3k_1 - 5k_2)\cos(t) = 2\sin(t).$$

This equation holds for all $t \in \mathbb{R}$. In particular, at $t = \frac{\pi}{2}$, $t = 0$.

$$\left.\begin{aligned} -5k_1 + 3k_2 &= 2, \\ -3k_1 - 5k_2 &= 0, \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} k_1 = -\dfrac{5}{17}, \\ k_2 = \dfrac{3}{17}. \end{cases}$$

Undetermined coefficients (2.5).

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So the particular solution to the inhomogeneous equation is

$$y_p(t) = \frac{1}{17}\left[-5\sin(t) + 3\cos(t)\right].$$

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The general solution is

$$y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{1}{17}\left[-5\sin(t) + 3\cos(t)\right].$$

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Undetermined coefficients (2.5)

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Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}$$

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$$y_p = x\big[k_1 \sin(2x) + k_2 \cos(2x)\big].$$

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$$4k_1 = 0, \quad -4k_2 = 3$$

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$$4k_1 = 0, \quad -4k_2 = 3 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = -\frac{3}{4}.$$

Therefore, $y_{p_1} = -\dfrac{3}{4}\, x\cos(2x).$

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We now compute y_{p_2} for $f_2(x) = e^{3x}$.

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We now compute y_{p_2} for $f_2(x) = e^{3x}$.

We guess: $y_{p_2} = k\, e^{3x}$. Then, $y''_{p_2} = 9\, e^{3x}$,

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$$(9+4)ke^{3x} = e^{3x} \quad \Rightarrow \quad k = \frac{1}{13} \quad \Rightarrow \quad y_{p_2} = \frac{1}{13}\, e^{3x}.$$

Therefore, the general solution is

$$y(x) = c_1 \sin(2x) + \Big(c_2 - \frac{3}{4}\, x\Big) \cos(2x) + \frac{1}{13}\, e^{3x}.$$

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Undetermined coefficients (2.5).

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$,

Undetermined coefficients (2.5).

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess

$$y_p(t) = \left[k_1 \sin(t) + k_2 \cos(t)\right] e^{2t}.$$

Undetermined coefficients (2.5).

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess

$$y_p(t) = \left[k_1 \sin(t) + k_2 \cos(t)\right] e^{2t}.$$

- For $y'' - 3y' - 4y = 2t^2\, e^{3t}$,

Undetermined coefficients (2.5).

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- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess
$$y_p(t) = \big[k_1 \sin(t) + k_2 \cos(t)\big]\, e^{2t}.$$
- For $y'' - 3y' - 4y = 2t^2\, e^{3t}$, guess
$$y_p(t) = \big(k_0 + k_1 t + k_2 t^2\big)\, e^{3t}.$$

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- For $y'' - 3y' - 4y = 3t\,\sin(t)$, guess
$$y_p(t) = (1 + k_1 t)\, \big[k_2 \sin(t) + k_3 \cos(t)\big].$$

Review for Exam 2.

- 6 or 7 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Variation of parameters (2.6).
  - Undetermined coefficients (2.5).
  - Constant coefficients, homogeneous, (2.2)-(2.4).
  - **Reduction order method, (2.4.2).**
  - Second order variable coefficients, (2.1).
  - First order homogeneous (1.3.2).

Reduction order method, (2.4.2).

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Reduction order method, (2.4.2).

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2t y' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Express $y_2(t) = v(t)\, y_1(t)$.

Reduction order method, (2.4.2).

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Solution: Express $y_2(t) = v(t)\, y_1(t)$. The equation for v comes from $t^2 y_2'' + 2ty_2' - 2y_2 = 0$.

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Solution: Express $y_2(t) = v(t)\, y_1(t)$. The equation for v comes from $t^2 y_2'' + 2ty_2' - 2y_2 = 0$. We need to compute

$$y_2 = v\,t,$$

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$$y_2 = v\,t, \qquad y_2' = t\,v' + v,$$

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$$y_2 = v\, t, \qquad y_2' = t\, v' + v, \qquad y_2'' = t\, v'' + 2v'.$$

So, the equation for v is given by

$$t^2 \big(t\, v'' + 2v' \big) + 2t \big(t\, v' + v \big) - 2t\, v = 0$$

Reduction order method, (2.4.2).

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Express $y_2(t) = v(t)\, y_1(t)$. The equation for v comes from $t^2 y_2'' + 2ty_2' - 2y_2 = 0$. We need to compute

$$y_2 = v\,t, \qquad y_2' = t\,v' + v, \qquad y_2'' = t\,v'' + 2v'.$$

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$$t^2\big(t\,v'' + 2v'\big) + 2t\big(t\,v' + v\big) - 2t\,v = 0$$

$$t^3\,v'' + (2t^2 + 2t^2)\,v' + (2t - 2t)\,v = 0$$

Reduction order method, (2.4.2).

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Express $y_2(t) = v(t)\, y_1(t)$. The equation for v comes from $t^2 y_2'' + 2ty_2' - 2y_2 = 0$. We need to compute

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$$t^3\,v'' + (2t^2 + 2t^2)\,v' + (2t - 2t)\,v = 0$$

$$t^3\,v'' + (4t^2)\,v' = 0$$

Reduction order method, (2.4.2).

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Express $y_2(t) = v(t)\, y_1(t)$. The equation for v comes from $t^2 y_2'' + 2ty_2' - 2y_2 = 0$. We need to compute

$$y_2 = v\,t, \qquad y_2' = t\,v' + v, \qquad y_2'' = t\,v'' + 2v'.$$

So, the equation for v is given by

$$t^2\big(t\,v'' + 2v'\big) + 2t\big(t\,v' + v\big) - 2t\,v = 0$$

$$t^3\,v'' + (2t^2 + 2t^2)\,v' + (2t - 2t)\,v = 0$$

$$t^3\,v'' + (4t^2)\,v' = 0 \quad \Rightarrow \quad v'' + \frac{4}{t}v' = 0.$$

Reduction order method, (2.4.2).

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

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Solution: Recall: $v'' + \frac{4}{t} v' = 0$.

This is a first order equation for $w = v'$, given by $w' + \frac{4}{t} w = 0$, so

$$\frac{w'}{w} = -\frac{4}{t}$$

Reduction order method, (2.4.2).

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knowing that $y_1(t) = t$ is a solution.

Solution: Recall: $v'' + \frac{4}{t} v' = 0$.

This is a first order equation for $w = v'$, given by $w' + \frac{4}{t} w = 0$, so

$$\frac{w'}{w} = -\frac{4}{t} \;\Rightarrow\; \ln(w) = -4\ln(t) + c_0$$

Reduction order method, (2.4.2).

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2t y' - 2y = 0,$$

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Recalling that $y_2 = t\, v$ we then conclude that $y_2 = c_2 t^{-2} + c_3 t$.

Reduction order method, (2.4.2).

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Recall: $v'' + \frac{4}{t} v' = 0$.

This is a first order equation for $w = v'$, given by $w' + \frac{4}{t} w = 0$, so

$$\frac{w'}{w} = -\frac{4}{t} \;\Rightarrow\; \ln(w) = -4\ln(t) + c_0 \;\Rightarrow\; w(t) = c_1 t^{-4},\; c_1 \in \mathbb{R}.$$

Integrating w we obtain v, that is, $v = c_2 t^{-3} + c_3$, with $c_2, c_3 \in \mathbb{R}$. Recalling that $y_2 = t\,v$ we then conclude that $y_2 = c_2 t^{-2} + c_3 t$. Choosing $c_2 = 1$ and $c_3 = 0$ we obtain the fundamental solutions

$y_1(t) = t$ and $y_2(t) = \frac{1}{t^2}$. ◁

Review for Exam 2.

- 6 or 7 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Variation of parameters (2.6).
  - Undetermined coefficients (2.5).
  - Constant coefficients, homogeneous, (2.2)-(2.4).
  - Reduction order method, (2.4.2).
  - Second order variable coefficients, (2.1).
  - **First order homogeneous (1.3.2).**

First order Homogeneous (1.3.2).

Example

Find all solutions y of the equation $y' = \dfrac{t^2 + 3y^2}{2ty}$.

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Solution: The equation is homogeneous, since

$$y' = \frac{t^2 + 3y^2}{2ty} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{1}{t^2}\right)}$$

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$$v + t\,v' = \frac{1 + 3v^2}{2v} \quad \Rightarrow \quad t\,v' = \frac{1 + 3v^2}{2v} - v$$

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We obtain the separable equation $v' = \dfrac{1}{t}\left(\dfrac{1 + v^2}{2v}\right)$.

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But $u = e^{\ln(t)}e^{c_0}$, so denoting $c_1 = e^{c_0}$, then $u = c_1 t$. Hence

$$1 + v^2 = c_1 t \quad \Rightarrow \quad 1 + \left(\frac{y}{t}\right)^2 = c_1 t \quad \Rightarrow \quad y(t) = \pm t\sqrt{c_1 t - 1}.$$

Mechanical and electrical oscillations (Sect. 2.7?)

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Application: Mechanical Oscillations.
- Application: The RLC electrical circuit.

Remark:
Different physical systems may have identical mathematical descriptions.

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Summary of solutions of the differential equation

$$y'' + a_1 y' + a_0 y = 0, \qquad a_1, a_2 \in \mathbb{R},$$

and characteristic roots $r_\pm = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0}$.

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$$y_1(t) = e^{\alpha t} \cos(\beta t), \qquad y_2(t) = e^{\alpha t} \sin(\beta t).$$

with $\alpha = -\frac{a_1}{2}$, $\beta = \frac{1}{2}\sqrt{4a_0 - a_1^2}$.

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Summary of solutions of the differential equation

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with $\alpha = -\frac{a_1}{2}$, $\beta = \frac{1}{2}\sqrt{4a_0 - a_1^2}$. Not damped: If $a_1 = 0$.

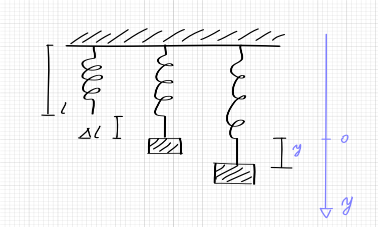
Mechanical and electrical oscillations (Sect. 2.7?)

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- **Application: Mechanical Oscillations.**
- Application: The RLC electrical circuit.

Application: Mechanical Oscillations.

Consider a spring attached to the ceiling, having rest length l, with an attached mass m.

- $(l + \Delta l)$ is called equilibrium position of the spring loaded with a mass m.
- The coordinate y measures vertical deviations from the equilibrium position.



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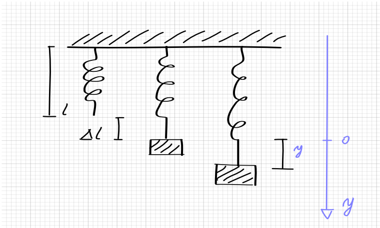

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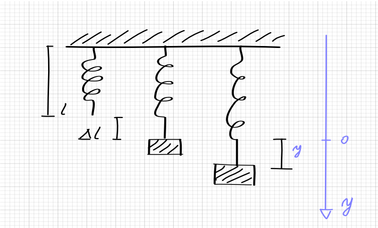

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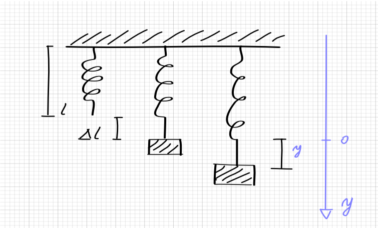

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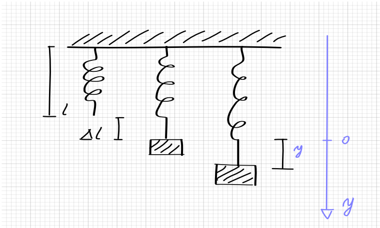

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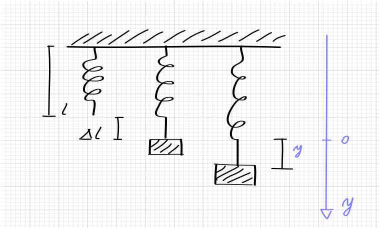

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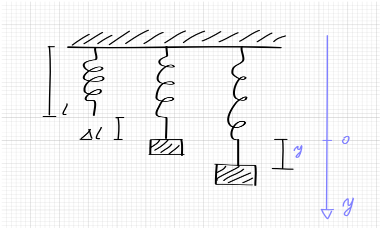

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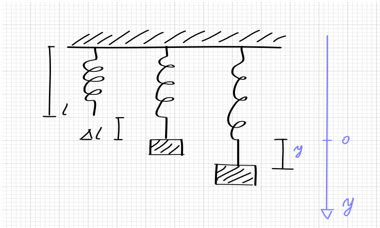

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Newton's Law: $m\, y''(t) = F_g + F_s(t) + F_d(t)$.

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□

Application: Mechanical Oscillations.

Damped Oscillations

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Application: Mechanical Oscillations.

Recall: $m\,y'' + d\,y' + k\,y = 0$, and $r_{\pm} = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}$.

(a) Over damped: $\omega_d > \omega_0$. Two distinct real roots:

$$y(t) = c_1 e^{r_+ t} + c_2\, e^{r_- t}.$$

(b) Critically damped: $\omega_d = \omega_0$. Repeated real root $r_+ = r_- = \hat{r}$:

$$y(t) = (c_1 + c_2 t)\, e^{\hat{r} t}.$$

(c) Under damped: $\omega_d < \omega_0$. Complex roots:

$$y(t) = \left[c_1 \cos(\beta t) + c_2 \sin(\beta t)\right] e^{-\omega_d t}$$

Application: Mechanical Oscillations.

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Application: Mechanical Oscillations.

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$$y(t) = A \cos(\beta t - \phi)\, e^{-\omega_d t}$$

where $r_{\pm} = -\omega_d \pm i\beta$, and $\beta = \sqrt{\omega_0^2 - \omega_d^2}$.

Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg/Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg/Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

Application: Mechanical Oscillations.

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Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg/Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg/Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

Solution: The equation is: $my'' + dy' + ky = 0$,

Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg/Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg/Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

Solution: The equation is: $my'' + dy' + ky = 0$, with $m = 5$, $k = 5$, $d = 5$.

Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg}/\text{Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg}/\text{Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

Solution: The equation is: $my'' + dy' + ky = 0$, with $m = 5$, $k = 5$, $d = 5$. The characteristic roots are

$$r_{\pm} = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2},$$

Application: Mechanical Oscillations.

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Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg}/\text{Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg}/\text{Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

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$$r_\pm = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}, \quad \omega_d = \frac{d}{2m}$$

Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg/Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg/Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

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Application: Mechanical Oscillations.

Example

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Application: Mechanical Oscillations.

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Application: Mechanical Oscillations.

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$$r_{\pm} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}$$

Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg/Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg/Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

Solution: The equation is: $my'' + dy' + ky = 0$, with $m = 5$, $k = 5$, $d = 5$. The characteristic roots are

$$r_{\pm} = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}, \quad \omega_d = \frac{d}{2m} = \frac{1}{2}, \quad \omega_0 = \sqrt{\frac{k}{m}} = 1.$$

$$r_{\pm} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1} = -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}.$$

Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg}/\text{Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg}/\text{Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

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$$r_{\pm} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1} = -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}. \quad \text{Under damped oscillations.}$$

Application: Mechanical Oscillations.

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Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg/Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg/Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

Solution: The equation is: $my'' + dy' + ky = 0$, with $m = 5$, $k = 5$, $d = 5$. The characteristic roots are

$$r_{\pm} = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}, \quad \omega_d = \frac{d}{2m} = \frac{1}{2}, \quad \omega_0 = \sqrt{\frac{k}{m}} = 1.$$

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$$y(t) = A\cos\Big(\frac{\sqrt{3}}{2}\,t - \phi\Big)\,e^{-t/2}.$$

Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg/Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg/Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

Solution: Recall: $y(t) = A\cos\Big(\dfrac{\sqrt{3}}{2}\,t - \phi\Big)\,e^{-t/2}$.

Application: Mechanical Oscillations.

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Solution: Recall: $y(t) = A\cos\Big(\frac{\sqrt{3}}{2}\,t - \phi\Big)\,e^{-t/2}$. Hence,

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Application: Mechanical Oscillations.

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The initial conditions:

$$\sqrt{3} = y(0)$$

Application: Mechanical Oscillations.

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$$\sqrt{3} = y(0) = A\cos(\phi),$$

Application: Mechanical Oscillations.

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Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg}/\text{Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg}/\text{Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

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Application: Mechanical Oscillations.

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$$\tan(\phi) = \frac{1}{\sqrt{3}}$$

Application: Mechanical Oscillations.

Example

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The initial conditions:

$$\sqrt{3} = y(0) = A\cos(\phi), \quad 0 = y'(0) = \frac{\sqrt{3}}{2}\,A\sin(\phi) - \frac{1}{2}\,A\cos(\phi).$$

$$\tan(\phi) = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \phi = \frac{\pi}{6},$$

Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg}/\text{Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg}/\text{Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

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Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg}/\text{Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg}/\text{Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

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$$\tan(\phi) = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \phi = \frac{\pi}{6}, \quad \Rightarrow \quad A = 2.$$

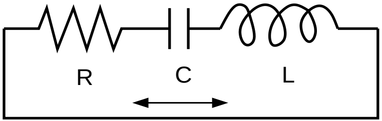
We conclude: $y(t) = 2\cos\Big(\frac{\sqrt{3}}{2}\,t - \frac{\pi}{6}\Big)\,e^{-t/2}$. ◁

Mechanical and electrical oscillations (Sect. 2.7?)

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Application: Mechanical Oscillations.
- **Application: The RLC electrical circuit.**

The RLC electrical circuit.

Consider an electric circuit with resistance R, non-zero capacitor C, and non-zero inductance L, as in the figure.



I (t) : electric current.

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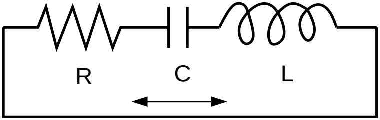

I (t) : electric current.

Kirchhoff's Law: The electric current flowing in the circuit satisfies:

$$L\,I'(t) + R\,I(t) + \frac{1}{C}\int_{t_0}^{t} I(s)\,ds = 0.$$

The RLC electrical circuit.

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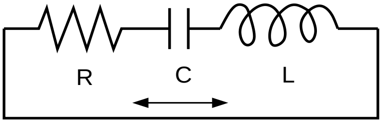

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$$L\,I'(t) + R\,I(t) + \frac{1}{C}\int_{t_0}^{t} I(s)\,ds = 0.$$

Derivate both sides above: $L\,I''(t) + R\,I'(t) + \dfrac{1}{C}\,I(t) = 0.$

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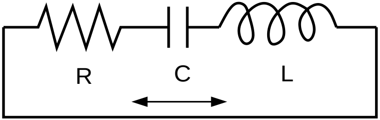

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Derivate both sides above: $L\, I''(t) + R\, I'(t) + \frac{1}{C}\, I(t) = 0.$

Divide by L: $I''(t) + 2\left(\frac{R}{2L}\right) I'(t) + \frac{1}{LC}\, I(t) = 0.$

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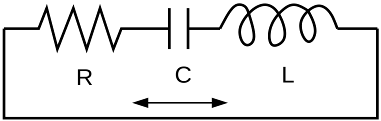
Consider an electric circuit with resistance R, non-zero capacitor C, and non-zero inductance L, as in the figure.



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Introduce $\alpha = \dfrac{R}{2L}$ and $\omega = \dfrac{1}{\sqrt{LC}}$,

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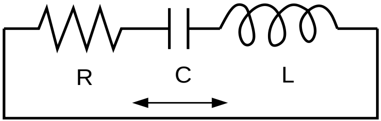
Consider an electric circuit with resistance R, non-zero capacitor C, and non-zero inductance L, as in the figure.



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Divide by L: $I''(t) + 2\Big(\dfrac{R}{2L}\Big)\, I'(t) + \dfrac{1}{LC}\, I(t) = 0.$

Introduce $\alpha = \dfrac{R}{2L}$ and $\omega = \dfrac{1}{\sqrt{LC}}$, then $I'' + 2\alpha\, I' + \omega^2\, I = 0.$

The RLC electrical circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha\, I' + \omega^2\, I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

The RLC electrical circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha\, I' + \omega^2\, I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$.

The RLC electrical circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha\, I' + \omega^2\, I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$. The roots are:

$$r_\pm = \frac{1}{2}\big[-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}\big]$$

The RLC electrical circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha\, I' + \omega^2\, I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

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Find real-valued fundamental solutions to $I'' + 2\alpha\, I' + \omega^2\, I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

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Case (a) $R = 0$.

The RLC electrical circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha\, I' + \omega^2\, I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

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Case (a) $R = 0$. This implies $\alpha = 0$,

The RLC electrical circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha\, I' + \omega^2\, I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$. The roots are:

$$r_\pm = \frac{1}{2}\big[-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}\big] \quad \Rightarrow \quad r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.$$

Case (a) $R = 0$. This implies $\alpha = 0$, so $r_\pm = \pm i\omega$.

The RLC electrical circuit.

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Remark: When the circuit has no resistance, the current oscillates without dissipation.

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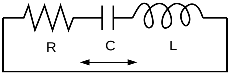

I (t) : electric current.

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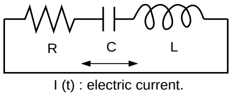

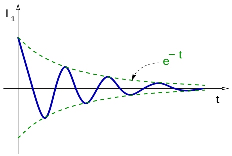

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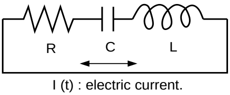

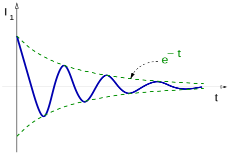

The resistance R damps the current oscillations.

The Euler equation (Sect. 3.2).

- We study the Euler Equation:
  $(x - x_0)^2 \, y'' + p_0 \, (x - x_0) \, y' + q_0 \, y = 0.$
- Solutions to the Euler equation near x_0.
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Given real constants p_0, q_0, the *Euler differential equation* for the unknown y with singular point at $x_0 \in R$ is given by

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- The point $x_0 \in \mathbb{R}$ is a singular point of the equation.
- The particular case $x_0 = 0$ is is given by

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- The main idea to find solution to the constant coefficients equation $y'' + a_1\, y' + a_0\, y = 0$ was to look for functions of the form $y(x) = e^{rx}$.

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  but the later equation still involves the variable x.

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The last equation involves only r, not x.

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This equation is called the indicial equation, and is also called the Euler characteristic equation.

Solutions to the Euler equation near x_0.

Theorem (Euler equation, $x_0 = 0$)

Given p_0, q_0, $x_0 \in \mathbb{R}$, consider the Euler equation

$$x^2\, y'' + p_0\, x\, y' + q_0\, y = 0. \tag{2}$$

Let r_+, r_- be solutions of $r(r-1) + p_0 r + q_0 = 0$.

(a) *If $r_+ \neq r_-$, then a general solution of Eq. (2) is*

$$y(x) = c_0 |x|^{r_+} + c_1 |x|^{r_-}, \quad x \neq 0, \quad c_0,\ c_1 \in \mathbb{R}\ (\text{or } \mathbb{C}).$$

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$$y(x) = \big[c_0 + c_1 \ln|x|\big]\, |x|^{\hat{r}}, \quad x \neq 0, \quad c_0,\ c_1 \in \mathbb{R}.$$

Given $x_1 \neq 0$, y_0, $y_1 \in \mathbb{R}$, there is a unique solution to the IVP

$$x^2\, y'' + p_0\, x\, y' + q_0\, y = 0, \quad y(x_1) = y_0, \quad y'(x_1) = y_1.$$

Solutions to the Euler equation near x_0.

Theorem (Euler equation, $x_0 \neq 0$)

Given p_0, q_0, $x_0 \in \mathbb{R}$, consider the Euler equation

$$(x - x_0)^2\, y'' + p_0\,(x - x_0)\,y' + q_0\,y = 0. \tag{3}$$

Let r_+, r_- be solutions of $r(r-1) + p_0 r + q_0 = 0$.

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Find the general solution of the Euler equation

$$x^2 \, y'' + 4x \, y' + 2 \, y = 0.$$

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$$x^2\, y'' + 4x\, y' + 2\, y = 0.$$

Solution: We look for solutions of the form $y(x) = x^r$,

$$x\, y'(x) = r x^r, \qquad x^2\, y''(x) = r(r-1)\, x^r.$$

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The solutions of the indicial equation $r^2 - 4r + 13 = 0$ are

$$r_{\pm} = \frac{1}{2}\big[4 \pm \sqrt{16 - 52}\big]$$

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The general solution is $y(x) = c_1\, |x|^{(2+3i)} + c_2\, |x|^{(2-3i)}$. $\lhd$

Different complex roots.

Theorem (Real-valued fundamental solutions)

If p_0, $q_0 \in \mathbb{R}$ satisfy that $\big[(p_0 - 1)^2 - 4q_0\big] < 0$, then the indicial polynomial $p(r) = r(r-1) + p_0 r + q_0$ of the Euler equation

$$x^2\, y'' + p_0 x\, y' + q_0\, y = 0 \tag{4}$$

has complex roots $r_+ = \alpha + i\beta$ and $r_- = \alpha - i\beta$, where

$$\alpha = -\frac{(p_0 - 1)}{2}, \qquad \beta = \frac{1}{2}\sqrt{4q_0 - (p_0 - 1)^2}.$$

A complex-valued fundamental set of solution to Eq. (4) is

$$\tilde{y}_1(x) = |x|^{(\alpha + i\beta)}, \qquad \tilde{y}_2(x) = |x|^{(\alpha - i\beta)}.$$

A real-valued fundamental set of solutions to Eq. (4) is

$$y_1(x) = |x|^{\alpha} \cos\big(\beta \ln|x|\big), \qquad y_2(x) = |x|^{\alpha} \sin\big(\beta \ln|x|\big).$$

Different complex roots.

Proof: Given $\tilde{y}_1 = |x|^{(\alpha+i\beta)}$ and $\tilde{y}_2 = |x|^{(\alpha-i\beta)}$,

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Use another Euler equation to rewrite $\tilde{y}_1$ and $\tilde{y}_2$,

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We conclude that

$$y_1(x) = |x|^{\alpha}\cos\big(\beta\ \ln|x|\big), \qquad y_2(x) = |x|^{\alpha}\sin\big(\beta\ \ln|x|\big).$$

$\square$

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Example

Find a real-valued general solution of the Euler equation

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$$y(x) = \tilde{c}_1\, |x|^{(2+3i)} + \tilde{c}_2\, |x|^{(2-3i)} \quad \tilde{c}_1,\ \tilde{c}_2 \in \mathbb{C}.$$

A real-valued general solution is

$$y(x) = c_1\, |x|^2 \cos\big(3 \ln |x|\big) + c_2\, |x|^2 \sin\big(3 \ln |x|\big), \quad c_1,\ c_2 \in \mathbb{R}. \quad \triangleleft$$