Non-homogeneous equations (Sect. 2.5).

- We study: $y'' + a_1\, y' + a_0\, y = b(t)$.
- Operator notation and preliminary results.
- Summary of the undetermined coefficients method.
- Using the method in few examples.
- The guessing solution table.

Operator notation and preliminary results.

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$$L(y) = y'' + p(t)\,y' + q(t)\,y.$$

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The function L acting on a function y is called an operator.

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Theorem

For every continuously differentiable functions y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ *and every* c_1, $c_2 \in \mathbb{R}$ *holds that*

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2).$$

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Proof:

$$L(c_1 y_1 + c_2 y_2) = (c_1 y_1 + c_2 y_2)'' + p(t)\,(c_1 y_1 + c_2 y_2)' + q(t)\,(c_1 y_1 + c_2 y_2)$$

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□

Operator notation and preliminary results.

Theorem

Given functions p, q, f, let $L(y) = y'' + p(t)\,y' + q(t)\,y$.
If the functions y_1 and y_2 are fundamental solutions of the homogeneous equation

$$L(y) = 0,$$

and y_p is any solution of the non-homogeneous equation

$$L(y_p) = f, \tag{1}$$

then any other solution y of the non-homogeneous equation above is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t), \tag{2}$$

where c_1, $c_2 \in \mathbb{R}$.

Operator notation and preliminary results.

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Given functions p, q, f, let $L(y) = y'' + p(t)\,y' + q(t)\,y$.
If the functions y_1 *and* y_2 *are fundamental solutions of the homogeneous equation*

$$L(y) = 0,$$

and y_p *is any solution of the non-homogeneous equation*

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then any other solution y of the non-homogeneous equation above is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t), \qquad (2)$$

where $c_1, c_2 \in \mathbb{R}$.

Notation: The expression for y in Eq. (2) is called the general solution of the non-homogeneous Eq. (1).

Operator notation and preliminary results.

Theorem

Given functions p, q, let $L(y) = y'' + p(t)\,y' + q(t)\,y$.
If the function f can be written as $f(t) = f_1(t) + \cdots + f_n(t)$, *with* $n \geqslant 1$, *and if there exist functions* $y_{p_1}, \cdots, y_{p_n}$ *such that*

$$L(y_{p_i}) = f_i, \qquad i = 1, \cdots, n,$$

then the function $y_p = y_{p_1} + \cdots + y_{p_n}$ *satisfies the non-homogeneous equation*

$$L(y_p) = f.$$

Non-homogeneous equations (Sect. 2.5).

- We study: $y'' + a_1\, y' + a_0\, y = b(t)$.
- Operator notation and preliminary results.
- **Summary of the undetermined coefficients method.**
- Using the method in few examples.
- The guessing solution table.

Summary of the undetermined coefficients method.

Problem: Given a constant coefficients linear operator $L(y) = y'' + a_1 y' + a_0 y$, with a_1, $a_2 \in \mathbb{R}$, find every solution of the non-homogeneous differential equation

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Remarks:

- The undetermined coefficients is a method to find solutions to linear, non-homogeneous, constant coefficients, differential equations.

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Remarks:

- The undetermined coefficients is a method to find solutions to linear, non-homogeneous, constant coefficients, differential equations.
- It consists in guessing the solution y_p of the non-homogeneous equation

  $$L(y_p) = f,$$

  for particularly simple source functions f.

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(2) If f has the form $f = f_1 + \cdots + f_n$, with $n \geqslant 1$, then look for solutions y_{p_i}, with $i = 1, \cdots, n$ to the equations

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Once the functions y_{p_i} are found, then construct

$$y_p = y_{p_1} + \cdots + y_{p_n}.$$

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$$y_p = y_{p_1} + \cdots + y_{p_n}.$$

(3) Given the source functions f_i, guess the solutions functions y_{p_i} following the Table below.

Summary of the undetermined coefficients method.

Summary (cont.):

$f_i(t)$ (K, m, a, b, given.)	$y_{p_i}(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
Kt^m	$k_m t^m + k_{m-1} t^{m-1} + \cdots + k_0$
$K\cos(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$K\sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$Kt^m e^{at}$	$e^{at}(k_m t^m + \cdots + k_0)$
$Ke^{at}\cos(bt)$	$e^{at}\big[k_1 \cos(bt) + k_2 \sin(bt)\big]$
$KKe^{at}\sin(bt)$	$e^{at}\big[k_1 \cos(bt) + k_2 \sin(bt)\big]$
$Kt^m \cos(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1 \cos(bt) + a_2 \sin(bt)\big]$
$Kt^m \sin(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1 \cos(bt) + a_2 \sin(bt)\big]$

Summary of the undetermined coefficients method.

Summary (cont.):

(4) If any guessed function y_{p_i} satisfies the homogeneous equation $L(y_{p_i}) = 0$, then change the guess to the function

$$t^s y_{p_i}, \quad \text{with} \quad s \geqslant 1,$$

and s sufficiently large such that $L(t^s y_{p_i}) \neq 0$.

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(5) Impose the equation $L(y_{p_i}) = f_i$ to find the undetermined constants $k_1, \cdots, k_m$, for the appropriate m, given in the table above.

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and s sufficiently large such that $L(t^s y_{p_i}) \neq 0$.

(5) Impose the equation $L(y_{p_i}) = f_i$ to find the undetermined constants $k_1, \cdots, k_m$, for the appropriate m, given in the table above.

(6) The general solution to the original differential equation $L(y) = f$ is then given by

$$y(t) = y_h(t) + y_{p_1} + \cdots + y_{p_n}.$$

Non-homogeneous equations (Sect. 2.5).

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Using the method in few examples.

Example

Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

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Solution: Notice: $L(y) = y'' - 3y' - 4y$ and $f(t) = 3e^{2t}$.

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The characteristic equation is

$$r^2 - 3r - 4 = 0$$

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$$r^2 - 3r - 4 = 0 \quad \Rightarrow \quad \begin{cases} r_1 = 4, \\ r_2 = -1. \end{cases}$$

$$y_h(t) = c_1\, e^{4t} + c_2\, e^{-t}.$$

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(2) Trivial in our case. The source function $f(t) = 3e^{2t}$ cannot be simplified into a sum of simpler functions.

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(3) Table says: For $f(t) = 3e^{2t}$ guess $y_p(t) = k\, e^{2t}$

Using the method in few examples.

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Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

Solution: Recall: $y_p(t) = k\, e^{2t}$. We need to find k.

Using the method in few examples.

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Solution: Recall: $y_p(t) = k\, e^{2t}$. We need to find k.

(4) Trivial here, since $L(y_p) \neq 0$, we do not modify our guess.

Using the method in few examples.

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(5) Introduce y_p into $L(y_p) = f$ and find k.

Using the method in few examples.

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$$(2^2 - 6 - 4)ke^{2t} = 3e^{2t}$$

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(5) Introduce y_p into $L(y_p) = f$ and find k.

$$(2^2 - 6 - 4)ke^{2t} = 3e^{2t} \quad \Rightarrow \quad -6k = 3$$

Using the method in few examples.

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We have obtained that $y_p(t) = -\frac{1}{2}\, e^{2t}$.

Using the method in few examples.

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We have obtained that $y_p(t) = -\frac{1}{2}\, e^{2t}$.

(6) The general solution to the inhomogeneous equation is

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}\, e^{2t}.$$

$\triangleleft$

Using the method in few examples.

Example

Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{4t}.$$

Using the method in few examples.

Example

Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{4t}.$$

Solution: We know that the general solution to homogeneous equation is $y_h(t) = c_1 e^{4t} + c_2 e^{-t}$.

Using the method in few examples.

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$$y'' - 3y' - 4y = 3e^{4t}.$$

Solution: We know that the general solution to homogeneous equation is $y_h(t) = c_1 e^{4t} + c_2 e^{-t}$.

Following the table we guess y_p as $y_p = k\, e^{4t}$.

Using the method in few examples.

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Solution: We know that the general solution to homogeneous equation is $y_h(t) = c_1 e^{4t} + c_2 e^{-t}$.

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However, this guess satisfies $L(y_p) = 0$.

Using the method in few examples.

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So we modify the guess to $y_p = kt\, e^{4t}$.

Using the method in few examples.

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Introduce the guess into $L(y_p) = f$.

Using the method in few examples.

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Introduce the guess into $L(y_p) = f$. We need to compute

$$y_p' = k\, e^{4t} + 4kt\, e^{4t}, \qquad y_p'' = 8k\, e^{4t} + 16kt\, e^{4t}.$$

Using the method in few examples.

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Find all solutions to the non-homogeneous equation

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$$\big[(8k + 16kt) - 3(k + 4kt) - 4kt\big]\, e^{4t} = 3e^{4t}.$$

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$$\big[(8k + 16kt) - 3(k + 4kt) - 4kt\big]\,e^{4t} = 3e^{4t}.$$

$$\big[(8+16t) - 3(1+4t) - 4t\big]\,k = 3$$

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$$\big[(8+16t) - 3(1+4t) - 4t\big]\, k = 3 \quad \Rightarrow \quad \big[5 + (16-12-4)\, t\big]\, k = 3$$

Using the method in few examples.

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Find all solutions to the non-homogeneous equation

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Solution: Recall:

$$y_p = kt\, e^{4t}, \quad y_p' = k\, e^{4t} + 4kt\, e^{4t}, \quad y_p'' = 8k\, e^{4t} + 16kt\, e^{4t}.$$

$$\big[(8k + 16kt) - 3(k + 4kt) - 4kt\big]\, e^{4t} = 3e^{4t}.$$

$$\big[(8+16t) - 3(1+4t) - 4t\big]\, k = 3 \quad \Rightarrow \quad \big[5 + (16-12-4)\, t\big]\, k = 3$$

We obtain that $k = \dfrac{3}{5}$.

Using the method in few examples.

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Find all solutions to the non-homogeneous equation

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Using the method in few examples.

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Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

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Compute: $y_p' = k_1 \cos(t) - k_2 \sin(t)$, $y_p'' = -k_1 \sin(t) - k_2 \cos(t)$.

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$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$

$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

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This equation holds for all $t \in \mathbb{R}$. In particular, at $t = \frac{\pi}{2}$, $t = 0$.

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$$\left.\begin{aligned} -5k_1 + 3k_2 &= 2, \\ -3k_1 - 5k_2 &= 0, \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} k_1 = -\dfrac{5}{17}, \\ k_2 = \dfrac{3}{17}. \end{cases}$$

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$$y_p(t) = \frac{1}{17}\left[-5\sin(t) + 3\cos(t)\right].$$

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We have just found out that

$$y_p(t) = -\frac{1}{2}\, e^{2t}, \qquad y_{p_2}(t) = \frac{1}{17}\left[-5\sin(t) + 3\cos(t)\right].$$

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We conclude that

$$y(t) = c_1 e^{4t} + c_2 e^{2t} - \frac{1}{2}\,e^{2t} + \frac{1}{17}\,\big[-5\sin(t) + 3\cos(t)\big]. \qquad \triangleleft$$

Using the method in few examples.

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$,

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- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess

$$y_p(t) = \left[k_1 \sin(t) + k_2 \cos(t)\right] e^{2t}.$$

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- For $y'' - 3y' - 4y = 3t\, \sin(t)$, guess

$$y_p(t) = (1 + k_1 t)\left[k_2 \sin(t) + k_3 \cos(t)\right].$$

Non-homogeneous equations (Sect. 2.5).

- We study: $y'' + a_1\, y' + a_0\, y = b(t)$.
- Operator notation and preliminary results.
- Summary of the undetermined coefficients method.
- Using the method in few examples.
- **The guessing solution table.**

The guessing solution table.

Guessing Solution Table.

$f_i(t)$ $(K, m, a, b,$ given.)	$y_{p_i}(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
Kt^m	$k_m t^m + k_{m-1} t^{m-1} + \cdots + k_0$
$K\cos(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$K\sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$Kt^m e^{at}$	$e^{at}(k_m t^m + \cdots + k_0)$
$Ke^{at}\cos(bt)$	$e^{at}\left[k_1 \cos(bt) + k_2 \sin(bt)\right]$
$KKe^{at}\sin(bt)$	$e^{at}\left[k_1 \cos(bt) + k_2 \sin(bt)\right]$
$Kt^m\cos(bt)$	$\left(k_m t^m + \cdots + k_0\right)\left[a_1 \cos(bt) + a_2 \sin(bt)\right]$
$Kt^m\sin(bt)$	$\left(k_m t^m + \cdots + k_0\right)\left[a_1 \cos(bt) + a_2 \sin(bt)\right]$

Non-homogeneous equations (Sect. 2.6).

- We study: $y'' + p(t)\,y' + q(t)\,y = f(t)$.
- Method of variation of parameters.
- Using the method in an example.
- The proof of the variation of parameter method.
- Using the method in another example.

Method of variation of parameters.

Remarks:

- This is a general method to find solutions to equations having variable coefficients and non-homogeneous with a continuous but otherwise arbitrary source function,

$$y'' + p(t)\, y' + q(t)\, y = f(t).$$

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- The variation of parameter method can be applied to more general equations than the undetermined coefficients method.

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- The variation of parameter method can be applied to more general equations than the undetermined coefficients method.

- The variation of parameter method usually takes more time to implement than the simpler method of undetermined coefficients.

Method of variation of parameters.

Theorem (Variation of parameters)

Let p, q, $f : (t_1, t_2) \to \mathbb{R}$ be continuous functions, then let functions y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ be linearly independent solutions to the homogeneous equation

$$y'' + p(t)\, y' + q(t)\, y = 0,$$

and let the function $W_{y_1 y_2}$ be the Wronskian of solutions y_1 and y_2. If the functions u_1 and u_2 are defined by

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{y_1 y_2}(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{y_1 y_2}(t)}\, dt,$$

then a particular solution y_p to the non-homogeneous differential equation $y'' + p(t)\, y' + q(t)\, y = f(t)$ is given by

$$y_p = u_1 y_1 + u_2 y_2.$$

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Using the method in an example.

Example

Find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = 2e^{t}.$$

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Solution:

First: Find fundamental solutions to the homogeneous equation.
The characteristic equation is

$$r^2 - 5r + 6 = 0$$

Using the method in an example.

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The characteristic equation is

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Hence, $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$. Compute their Wronskian,

$$W_{y_1 y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t})$$

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$$W_{y_1 y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t}) \quad \Rightarrow \quad W_{y_1 y_2}(t) = -e^{5t}.$$

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Second: We compute the functions u_1 and u_2.

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$$W_{y_1y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t}) \quad \Rightarrow \quad W_{y_1y_2}(t) = -e^{5t}.$$

Second: We compute the functions u_1 and u_2. By definition,

$$u_1' = -\frac{y_2 f}{W_{y_1y_2}}, \qquad u_2' = \frac{y_1 f}{W_{y_1y_2}}.$$

Using the method in an example.

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$$y'' - 5y' + 6y = 2e^{t}.$$

Solution: Recall: $y_1(t) = e^{3t}$, $y_2(t) = e^{2t}$, $W_{y_1 y_2}(t) = -e^{5t}$, and

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Using the method in an example.

Example

Find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = 2e^t.$$

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Third: The particular solution is

$$y_p = (-e^{-2t})(e^{3t}) + (2e^{-t})(e^{2t})$$

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Third: The particular solution is

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The general solution is $y(t) = c_1 e^{3t} + c_2 e^{2t} + e^{t}$, $c_1, c_2 \in \mathbb{R}$. $\triangleleft$

Non-homogeneous equations (Sect. 2.6).

- We study: $y'' + p(t)\, y' + q(t)\, y = f(t)$.
- Method of variation of parameters.
- Using the method in an example.
- **The proof of the variation of parameter method.**
- Using the method in another example.

The proof of the variation of parameter method.

Proof: Denote $L(y) = y'' + p(t)\,y' + q(t)\,y$.

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First idea: Propose that y_p is given by $y_p = u_1 y_1 + u_2 y_2$.

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$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2',$$

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$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2',$$

$$y_p'' = u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''.$$

The proof of the variation of parameter method.

Proof: Then $L(y_p) = f$ is given by

$$\left[u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''\right]$$

$$p(t)\left[u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'\right] + q(t)\left[u_1 y_1 + u_2 y_2\right] = f(t).$$

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$$u_1'' y_1 + u_2'' y_2 + 2(u_1' y_1' + u_2 y_2') + p\,(u_1' y_1 + u_2' y_2)$$

$$+u_1\,(y_1'' + p\,y_1' + q\,y_1) + u_2\,(y_2'' + p\,y_2' + q\,y_2) = f$$

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Recall: $y_1'' + p\,y_1' + q\,y_1 = 0$ and $y_2'' + p\,y_2' + q\,y_2 = 0$.

The proof of the variation of parameter method.

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Second idea: Look for u_1 and u_2 that satisfy the extra equation

$$u_1' y_1 + u_2' y_2 = 0.$$

The proof of the variation of parameter method.

Proof: Recall: $u_1' y_1 + u_2' y_2 = 0$ and

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These two equations imply that $L(y_p) = f$ is

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This information in $L(y_p) = f$ implies

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From $u_1' y_1 + u_2' y_2 = 0$ we get $\left[u_1' y_1 + u_2' y_2\right]' = 0$, that is

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This information in $L(y_p) = f$ implies

$$u_1' y_1' + u_2' y_2' = f.$$

Summary: If u_1 and u_2 satisfy $u_1' y_1 + u_2' y_2 = 0$ and $u_1' y_1' + u_2' y_2' = f$, then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$.

The proof of the variation of parameter method.

Proof: Summary: If u_1 and u_2 satisfy $\left\{\begin{array}{l} u_1' y_1 + u_2' y_2 = 0, \\ u_1' y_1' + u_2' y_2' = f, \end{array}\right\}$ then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$.

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Since $W_{y_1 y_2} = y_1 y_2' - y_1' y_2$, then

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$$u_2' = -\frac{y_1}{y_2}\, u_1' \quad \Rightarrow \quad u_1' y_1' - \frac{y_1 y_2'}{y_2}\, u_1' = f \quad \Rightarrow \quad u_1' \Big(\frac{y_1' y_2 - y_1 y_2'}{y_2} \Big) = f.$$

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Since $W_{y_1 y_2} = y_1 y_2' - y_1' y_2$, then $\;u_1' = -\dfrac{y_2 f}{W_{y_1 y_2}} \quad \Rightarrow \quad u_2' = \dfrac{y_1 f}{W_{y_1 y_2}}.$

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Since $W_{y_1 y_2} = y_1 y_2' - y_1' y_2$, then $u_1' = -\dfrac{y_2 f}{W_{y_1 y_2}} \quad \Rightarrow \quad u_2' = \dfrac{y_1 f}{W_{y_1 y_2}}$.

Integrating in the variable t we obtain

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{y_1 y_2}(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{y_1 y_2}(t)}\, dt,$$

This establishes the Theorem. $\square$

Non-homogeneous equations (Sect. 2.6).

- We study: $y'' + p(t)\,y' + q(t)\,y = f(t)$.
- Method of variation of parameters.
- Using the method in an example.
- The proof of the variation of parameter method.
- **Using the method in another example.**

Using the method in another example.

Example

Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Using the method in another example.

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$$y'' - \frac{2}{t^2}\, y = 3 - \frac{1}{t^2} \quad \Rightarrow \quad f(t) = 3 - \frac{1}{t^2}.$$

Using the method in another example.

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Using the method in another example.

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We know that $y_1 = t^2$ and $y_2 = 1/t$. Their Wronskian is

$$W_{y_1 y_2}(t) = (t^2)\Big(\frac{-1}{t^2}\Big) - (2t)\Big(\frac{1}{t}\Big)$$

Using the method in another example.

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Solution: $y_1 = t^2$, $y_2 = 1/t$, $f(t) = 3 - \frac{1}{t^2}$, $W_{y_1 y_2}(t) = -3$.

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We now compute y_1 and u_2,

$$u_1' = -\frac{1}{t}\left(3 - \frac{1}{t^2}\right)\frac{1}{-3}$$

Using the method in another example.

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Using the method in another example.

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$$u_2' = (t^2)\left(3 - \frac{1}{t^2}\right)\frac{1}{-3} = -t^2 + \frac{1}{3} \quad \Rightarrow \quad u_2 = -\frac{1}{3}\,t^3 + \frac{1}{3}t.$$

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Solution: The particular solution $\tilde{y}_p = u_1 y_1 + u_2 y_2$ is

$$\tilde{y}_p = \left[\ln(t) + \frac{1}{6} t^{-2}\right](t^2) + \frac{1}{3}(-t^3 + t)(t^{-1})$$

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$$\tilde{y}_p = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\,y_1(t).$$

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$$\tilde{y}_p = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3} y_1(t).$$

A simpler expression is $y_p = t^2 \ln(t) + \frac{1}{2}$. ◁

Using the method in another example.

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Solution: If we do not remember the formulas for u_1, u_2, we can always solve the system

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f. \end{aligned}$$

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$$t^2 \, u_1' + u_2' \frac{1}{t} = 0, \quad 2t \, u_1' + u_2' \frac{(-1)}{t^2} = 3 - \frac{1}{t^2}.$$

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$$u_2' = -t^3\, u_1'$$

Using the method in another example.

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$$t^2\, u_1' + u_2' \frac{1}{t} = 0, \quad 2t\, u_1' + u_2' \frac{(-1)}{t^2} = 3 - \frac{1}{t^2}.$$

$$u_2' = -t^3\, u_1' \;\Rightarrow\; 2t\, u_1' + t\, u_1' = 3 - \frac{1}{t^2} \;\Rightarrow\; \begin{cases} u_1' = \dfrac{1}{t} - \dfrac{1}{3t^3} \\ u_2' = -t^2 + \dfrac{1}{3}. \end{cases}$$

Review for Exam 2.

- 6 or 7 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Variation of parameters (2.6).
  - Undetermined coefficients (2.5).
  - Constant coefficients, homogeneous, (2.2)-(2.4).
  - Reduction order method, (2.4.2).
  - Second order variable coefficients, (2.1).
  - First order homogeneous (1.3.2).

Review for Exam 2.

- 5 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - **Variation of parameters (2.6).**
  - Undetermined coefficients (2.5).
  - Constant coefficients, homogeneous, (2.2)-(2.4).
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  - Second order variable coefficients, (2.1).
  - First order homogeneous (1.3.2).

Variation of parameters (2.6).

Theorem (Variation of parameters)

Let p, q, $f : (t_1, t_2) \to \mathbb{R}$ be continuous functions, then let functions y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ be linearly independent solutions to the homogeneous equation

$$y'' + p(t)\, y' + q(t)\, y = 0,$$

and let the function $W_{y_1 y_2}$ be the Wronskian of solutions y_1 and y_2. If the functions u_1 and u_2 are defined by

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{y_1 y_2}(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{y_1 y_2}(t)}\, dt,$$

then a particular solution y_p to the non-homogeneous differential equation $y'' + p(t)\, y' + q(t)\, y = f(t)$ is given by

$$y_p = u_1 y_1 + u_2 y_2.$$

Variation of parameters (2.6).

Proof: Summary: If u_1 and u_2 satisfy $\left\{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0, \\ u_1' y_1' + u_2' y_2' &= f, \end{aligned}\right\}$ then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$.

Variation of parameters (2.6).

Proof: Summary: If u_1 and u_2 satisfy $\left\{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0, \\ u_1' y_1' + u_2' y_2' &= f, \end{aligned}\right\}$ then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$.

The equations above are simple to solve for u_1 and u_2,

Variation of parameters (2.6).

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The equations above are simple to solve for u_1 and u_2,

$$u_2' = -\frac{y_1}{y_2}\, u_1'$$

Variation of parameters (2.6).

Proof: Summary: If u_1 and u_2 satisfy $\left\{ \begin{array}{l} u_1' y_1 + u_2' y_2 = 0, \\ u_1' y_1' + u_2' y_2' = f, \end{array} \right\}$ then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$.

The equations above are simple to solve for u_1 and u_2,

$$u_2' = -\frac{y_1}{y_2}\, u_1' \quad \Rightarrow \quad u_1' y_1' - \frac{y_1 y_2'}{y_2}\, u_1' = f$$

Variation of parameters (2.6).

Proof: Summary: If u_1 and u_2 satisfy $\begin{cases} u_1' y_1 + u_2' y_2 = 0, \\ u_1' y_1' + u_2' y_2' = f, \end{cases}$ then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$.

The equations above are simple to solve for u_1 and u_2,

$$u_2' = -\frac{y_1}{y_2}\, u_1' \quad \Rightarrow \quad u_1' y_1' - \frac{y_1 y_2'}{y_2}\, u_1' = f \quad \Rightarrow \quad u_1' \Big(\frac{y_1' y_2 - y_1 y_2'}{y_2} \Big) = f.$$

Variation of parameters (2.6).

Proof: Summary: If u_1 and u_2 satisfy $\left\{\begin{array}{l} u_1' y_1 + u_2' y_2 = 0, \\ u_1' y_1' + u_2' y_2' = f, \end{array}\right\}$ then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$.

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Since $W_{y_1 y_2} = y_1 y_2' - y_1' y_2$, then

Variation of parameters (2.6).

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Since $W_{y_1 y_2} = y_1 y_2' - y_1' y_2$, then $u_1' = -\dfrac{y_2 f}{W_{y_1 y_2}}$

Variation of parameters (2.6).

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Since $W_{y_1 y_2} = y_1 y_2' - y_1' y_2$, then $u_1' = -\dfrac{y_2 f}{W_{y_1 y_2}} \quad \Rightarrow \quad u_2' = \dfrac{y_1 f}{W_{y_1 y_2}}.$

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Since $W_{y_1 y_2} = y_1 y_2' - y_1' y_2$, then $u_1' = -\dfrac{y_2 f}{W_{y_1 y_2}} \quad \Rightarrow \quad u_2' = \dfrac{y_1 f}{W_{y_1 y_2}}$.

Integrating in the variable t we obtain

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{y_1 y_2}(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{y_1 y_2}(t)}\, dt,$$

This establishes the Theorem. □

Variation of parameters (2.6).

Example

Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

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$$W_{y_1 y_2}(t) = (t^2)\Big(\frac{-1}{t^2}\Big) - (2t)\Big(\frac{1}{t}\Big)$$

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We now compute y_1 and u_2,

$$u_1' = -\frac{1}{t}\left(3 - \frac{1}{t^2}\right)\frac{1}{-3}$$

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A simpler expression is $y_p = t^2 \ln(t) + \frac{1}{2}$. ◁

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Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2}\, e^{-2x}.$$

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$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

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Hence $W = e^{-4x}$.

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$$y'' + 4y' + 4y = x^{-2}\, e^{-2x}.$$

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$$u_1' = -\frac{y_2 g}{W} = -\frac{x\, e^{-2x}\, x^{-2}\, e{-}2x}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln|x|.$$

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$$u_1' = -\frac{y_2 g}{W} = -\frac{x\, e^{-2x}\, x^{-2}\, e{-}2x}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln|x|.$$

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Variation of parameters (2.6).

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Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2}\,e^{-2x}.$$

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Since $\tilde{y}_p = -\ln|x|\, e^{-2x}$ is solution, $y = (c_1 + c_2 x - \ln|x|)\, e^{-2x}$. $\triangleleft$

Review for Exam 2.

- 5 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Variation of parameters (2.6).
  - **Undetermined coefficients (2.5).**
  - Constant coefficients, homogeneous, (2.2)-(2.4).
  - Reduction order method, (2.4.2).
  - Second order variable coefficients, (2.1).
  - First order homogeneous (1.3.2).

Undetermined coefficients (2.5).

Guessing Solution Table.

$f_i(t)$ $(K,\ m,\ a,\ b,$ given.)	$y_{p_i}(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
Kt^m	$k_m t^m + k_{m-1} t^{m-1} + \cdots + k_0$
$K\cos(bt)$	$k_1\cos(bt) + k_2\sin(bt)$
$K\sin(bt)$	$k_1\cos(bt) + k_2\sin(bt)$
$Kt^m e^{at}$	$e^{at}(k_m t^m + \cdots + k_0)$
$Ke^{at}\cos(bt)$	$e^{at}\big[k_1\cos(bt) + k_2\sin(bt)\big]$
$KKe^{at}\sin(bt)$	$e^{at}\big[k_1\cos(bt) + k_2\sin(bt)\big]$
$Kt^m\cos(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1\cos(bt) + a_2\sin(bt)\big]$
$Kt^m\sin(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1\cos(bt) + a_2\sin(bt)\big]$

Undetermined coefficients (2.5).

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

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Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: We know that the general solution to homogeneous equation is $y(t) = c_1 e^{4t} + c_2 e^{-t}$.

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$$y_p = k_1 \sin(t) + k_2 \cos(t).$$

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Compute: $y_p' = k_1 \cos(t) - k_2 \sin(t)$, $y_p'' = -k_1 \sin(t) - k_2 \cos(t)$.

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$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$

$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

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This equation holds for all $t \in \mathbb{R}$. In particular, at $t = \frac{\pi}{2}$, $t = 0$.

$$-5k_1 + 3k_2 = 2,$$
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$$\left.\begin{aligned} -5k_1 + 3k_2 &= 2, \\ -3k_1 - 5k_2 &= 0, \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} k_1 &= -\frac{5}{17}, \\ k_2 &= \frac{3}{17}. \end{aligned}\right.$$

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$$y'' + 4y = 3\sin(2x) + e^{3x}.$$

Solution: Recall: $y_1 = \sin(2x)$, and $y_2 = \cos(2x)$.

$$4\big[k_1\cos(2x) - k_2\sin(2x)\big] + 4x\big[-k_1\sin(2x) - k_2\cos(2x)\big] + 4x\big[k_1\sin(2x) + k_2\cos(2x)\big] = 3\sin(2x),$$

Therefore, $4\big[k_1\cos(2x) - k_2\sin(2x)\big] = 3\sin(2x)$.

Evaluating at $x = 0$ and $x = \pi/4$ we get

$$4k_1 = 0, \quad -4k_2 = 3 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = -\frac{3}{4}.$$

Therefore, $y_{p_1} = -\frac{3}{4}\,x\cos(2x)$.

Undetermined coefficients (2.5)

Example

Use the undetermined coefficients to find the general solution of

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$$(9+4)ke^{3x} = e^{3x} \quad \Rightarrow \quad k = \frac{1}{13} \quad \Rightarrow \quad y_{p_2} = \frac{1}{13}\, e^{3x}.$$

Therefore, the general solution is

$$y(x) = c_1 \sin(2x) + \left(c_2 - \frac{3}{4}\, x\right) \cos(2x) + \frac{1}{13}\, e^{3x}. \qquad \triangleleft$$

Undetermined coefficients (2.5).

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$,

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Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess

$$y_p(t) = \big[k_1 \sin(t) + k_2 \cos(t)\big]\, e^{2t}.$$

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Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess
$$y_p(t) = \left[k_1 \sin(t) + k_2 \cos(t)\right] e^{2t}.$$
- For $y'' - 3y' - 4y = 2t^2\, e^{3t}$, guess
$$y_p(t) = \left(k_0 + k_1 t + k_2 t^2\right) e^{3t}.$$

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$$y_p(t) = \left[k_1 \sin(t) + k_2 \cos(t)\right] e^{2t}.$$

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- For $y'' - 3y' - 4y = 3t\,\sin(t)$, guess

$$y_p(t) = (1 + k_1 t)\,\big[k_2 \sin(t) + k_3 \cos(t)\big].$$