Variable coefficients second order linear ODE (Sect. 2.1).

- Second order linear ODE.
- Superposition property.
- Existence and uniqueness of solutions.
- Linearly dependent and independent functions.
- The Wronskian of two functions.
- General and fundamental solutions.
- Abel's theorem on the Wronskian.
- Special Second order nonlinear equations.

Second order linear differential equations.

Definition

Given functions a_1, a_0, $b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t) \tag{1}$$

is called a *second order linear* differential equation with *variable coefficients*.

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Remark: The notion of an homogeneous equation presented here is not the same as the notion presented in the previous chapter.

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Example

(a) A second order, linear, homogeneous, constant coefficients equation is

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(d) Newton's second law of motion $(ma = f)$ for point particles of mass m moving in one space dimension under a force $f : \mathbb{R} \to \mathbb{R}$ is given by

$$m\,y''(t) = f(t). \qquad \triangleleft$$

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Theorem

If the functions y_1 *and* y_2 *are solutions to the homogeneous linear equation*

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0, \tag{2}$$

then the linear combination $c_1 y_1(t) + c_2 y_2(t)$ *is also a solution for any constants* c_1, $c_2 \in \mathbb{R}$.

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$$= c_1 \big[y_1'' + a_1(t) y_1' + a_0(t) y_1 \big] + c_2 \big[y_2'' + a_1(t) y_2' + a_0(t) y_2 \big] = 0.$$

$\square$

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Existence and uniqueness of solutions.

Theorem (Variable coefficients)

If the functions $a, b : (t_1, t_2) \to \mathbb{R}$ *are continuous, the constants* $t_0 \in (t_1, t_2)$ *and* y_0, $y_1 \in \mathbb{R}$, *then there exists a unique solution* $y : (t_1, t_2) \to \mathbb{R}$ *to the initial value problem*

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- Unlike the first order linear ODE where we have an explicit expression for the solution, there is no explicit expression for the solution of second order linear ODE.
- Two integrations must be done to find solutions to second order linear. Therefore, initial value problems with two initial conditions can have a unique solution.

Existence and uniqueness of solutions.

Example

Find the longest interval $I \in \mathbb{R}$ such that there exists a unique solution to the initial value problem

$$(t-1)y'' - 3ty' + 4y = t(t-1), \qquad y(-2) = 2, \qquad y'(-2) = 1.$$

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Definition

Two continuous functions $y_1,\ y_2 : (t_1, t_2) \subset \mathbb{R} \to \mathbb{R}$ are called *linearly dependent, (ld),* on the interval (t_1, t_2) iff there exists a constant c such that for all $t \in I$ holds

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The two functions are called *linearly independent, (li),* on the interval (t_1, t_2) iff they are not linearly dependent.

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- $y_1,\ y_2 : (t_1, t_2) \to \mathbb{R}$ are li $\Leftrightarrow$ the only constants $c_1,\ c_2$, solutions of $c_1\, y_1(t) + c_2\, y_2(t) = 0$ for all $t \in (t_1, t_2)$ are $c_1 = c_2 = 0$.

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Example

(a) Show that $y_1(t) = \sin(t)$, $y_2(t) = 2\sin(t)$ are ld.

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We conclude: The functions y_1 and y_2 are li. $\triangleleft$

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- If $A(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$, then $W_{y_1 y_2}(t) = \det\big(A(t)\big)$.

The Wronskian of two functions.

Remark: The Wronskian is a function that determines whether two functions are ld or li.

Definition

The *Wronskian* of functions y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ is the function

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- If $A(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$, then $W_{y_1 y_2}(t) = \det\big(A(t)\big)$.
- An alternative notation is: $W_{y_1 y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

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Example

Find the Wronskian of the functions:

(a) $y_1(t) = \sin(t)$ and $y_2(t) = 2\sin(t)$. (ld)

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We obtain $W_{y_1 y_2}(t) = \sin^2(t)$. $\triangleleft$

The Wronskian of two functions.

Remark: The Wronskian determines whether two functions are linearly dependent or independent.

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Theorem (Wronskian and linearly dependence)

The continuously differentiable functions y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ *are linearly dependent iff* $W_{y_1 y_2}(t) = 0$ *for all* $t \in (t_1, t_2)$.

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Remark: Importance of the Wronskian:

- Sometimes it is not simple to decide whether two functions are proportional to each other.
- The Wronskian is useful to study properties of solutions to ODE without having the explicit expressions of these solutions. (See Abel's Theorem later on.)

The Wronskian of two functions.

Example

Show whether the following two functions form a l.d. or l.i. set:

$$y_1(t) = \cos(2t) - 2\cos^2(t), \qquad y_2(t) = \cos(2t) + 2\sin^2(t).$$

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We conclude $W_{y_1 y_2}(t) = 0$, so the functions y_1 and y_2 are ld. $\triangleleft$

Variable coefficients second order linear ODE (Sect. 2.1).

- Second order linear ODE.
- Superposition property.
- Existence and uniqueness of solutions.
- Linearly dependent and independent functions.
- The Wronskian of two functions.
- **General and fundamental solutions.**
- Abel's theorem on the Wronskian.
- Special Second order nonlinear equations.

General and fundamental solutions.

Theorem

If a_1, $a_0 : (t_1, t_2) \to \mathbb{R}$ are continuous, then the functions $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ solutions of the initial value problems

$$
\begin{aligned}
y_1'' + a_1(t)\, y_1' + a_0(t)\, y_1 = 0, \qquad & y_1(0) = 1, \quad y_1'(0) = 0, \\
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Remarks:

- Every linear combination $y(t) = c_1\, y_1(t) + c_2\, y_2(t)$, is also a solution of the differential equation

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- Conversely, every solution y of the equation above can be written as a linear combination of the solutions y_1, y_2.

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Remark: The results above justify the following definitions.

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Definition
Two solutions y_1, y_2 of the homogeneous equation

$$y'' + a_1(t)y' + a_0(t)y = 0, \tag{3}$$

are called *fundamental solutions* iff the functions y_1, y_2 are linearly independent, that is, iff $W_{y_1 y_2} \neq 0$.

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Definition

Given any two fundamental solutions y_1, y_2, and arbitrary constants c_1, c_2, the function

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$$

is called the *general solution* of Eq. (3).

General and fundamental solutions.

Example

Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are fundamental solutions of

$$2t^2\,y'' + 3t\,y' - y = 0.$$

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Now show that y_2 is a solution:

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General and fundamental solutions.

Example

Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are fundamental solutions of

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Variable coefficients second order linear ODE (Sect. 2.1).

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Abel's theorem on the Wronskian.

Theorem (Abel)

If a_1, $a_0 : (t_1, t_2) \to \mathbb{R}$ are continuous functions and y_1, y_2 are continuously differentiable solutions of the equation

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then the Wronskian $W_{y_1 y_2}$ is a solution of the equation

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Therefore, for any $t_0 \in (t_1, t_2)$, the Wronskian $W_{y_1 y_2}$ is given by

$$W_{y_1 y_2}(t) = W_{y_1 y_2}(t_0)\, e^{A(t)} \qquad A(t) = \int_{t_0}^{t} a_1(s)\, ds.$$

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Remarks: If the the Wronskian of two solutions vanishes at the initial time, then it vanishes at all times.

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Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

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$$y'' - \Big(\frac{2}{t} + 1\Big)\, y' + \Big(\frac{2}{t^2} + \frac{1}{t}\Big)\, y = 0.$$

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Special Second order nonlinear equations

Definition

Given a functions $f : \mathbb{R}^3 \to \mathbb{R}$, *a second order* differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ is given by

$$y'' = f(t, y, y').$$

The equation is *linear* iff f is linear in the arguments y and y'.

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Remark: If second order differential equation has the form $y'' = f(t, y')$, then the equation for $v = y'$ is the first order equation $v' = f(t, v)$.

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We conclude $y = \frac{1}{2}\big(\ln|t-1| - \ln|t+1|\big) + 2$. $\triangleleft$

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Therefore, $\dfrac{d\hat{v}}{dy} = \dfrac{1}{\hat{v}} f(y, \hat{v}(y))$. □

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Again, for no reason, we choose $c_0 = 0$, and we conclude that one possible solution to our problem is $y(t) = \tan(t)$. $\triangleleft$

Second order linear ODE (Sect. 2.2).

- Review: Second order linear differential equations.
- Idea: Soving constant coefficients equations.
- The characteristic equation.
- Solution formulas for constant coefficients equations.

Review: Second order linear ODE.

Definition

Given functions a_1, a_0, $b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

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Theorem (Superposition property)

If the functions y_1 and y_2 are solutions to the homogeneous linear equation

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0,$$

then the linear combination $c_1 y_1(t) + c_2 y_2(t)$ is also a solution for any constants c_1, $c_2 \in \mathbb{R}$.

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Find solutions to the equation $y'' + 5y' + 6y = 0$.

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Solution: We look for solutions proportional to exponentials e^{rt}, for an appropriate constant $r \in \mathbb{R}$, since the exponential can be canceled out from the equation.

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This polynomial is called the characteristic polynomial of the differential equation.

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Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: Recall: $p(r) = r^2 + 5r + 6$.

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$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

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Summary: The differential equation $y'' + 5y' + 6y = 0$ has infinitely many solutions,

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- An IVP for a second order differential equation will have a unique solution if the IVP contains two initial conditions.

Second order linear ODE (Sect. 2.2).

- Review: Second order linear differential equations.
- Idea: Soving constant coefficients equations.
- **The characteristic equation.**
- Solution formulas for constant coefficients equations.

The characteristic equation.

Definition

Given a second order linear homogeneous differential equation with constant coefficients

$$y'' + a_1 y' + a_0 = 0, \tag{4}$$

the *characteristic polynomial* and the *characteristic equation* associated with the differential equation in (4) are, respectively,

$$p(r) = r^2 + a_1 r + a_0, \qquad p(r) = 0.$$

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Remark: If r_1, r_2 are the solutions of the characteristic equation and c_1, c_2 are constants, then we will show that the general solution of Eq. (4) is given by

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

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Example

Find the solution y of the initial value problem

$$y'' + 5y' + 6 = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

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Therefore, the unique solution to the initial value problem is

$$y(t) = 2e^{-2t} - e^{-3t}.$$

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Therefore, the general solution of the equation above is

$$y(t) = c_1 e^{t} + c_2 e^{t/2},$$

where c_1, c_2 are arbitrary constants. $\triangleleft$

Second order linear ODE (Sect. 2.2).

- Review: Second order linear differential equations.
- Idea: Soving constant coefficients equations.
- The characteristic equation.
- **Solution formulas for constant coefficients equations.**

Solution formulas for constant coefficients equations.

Theorem (Constant coefficients)

Given real constants a_1, a_0, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \to \mathbb{R}$ given by

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Furthermore, given real constants t_0, y_0 and y_1, there is a unique solution to the initial value problem

$$y'' + a_1\, y' + a_0\, y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_1.$$