Modeling with first order equations (Sect. 1.5).

- Radioactive decay.
  - Carbon-14 dating.
- Salt in a water tank.
  - The experimental device.
  - The main equations.
  - Analysis of the mathematical model.
  - Predictions for particular situations.

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(e) Using the half-life, we get $N(t) = N_0\,2^{-t/\tau}$.

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The organism died 16,253 years ago. ◁

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- To construct a model means to find the differential equation that takes into account the above properties of the system.
- Finding the solution to the differential equation with a particular initial condition means we can predict the evolution of the salt in the tank if we know the tank initial condition.

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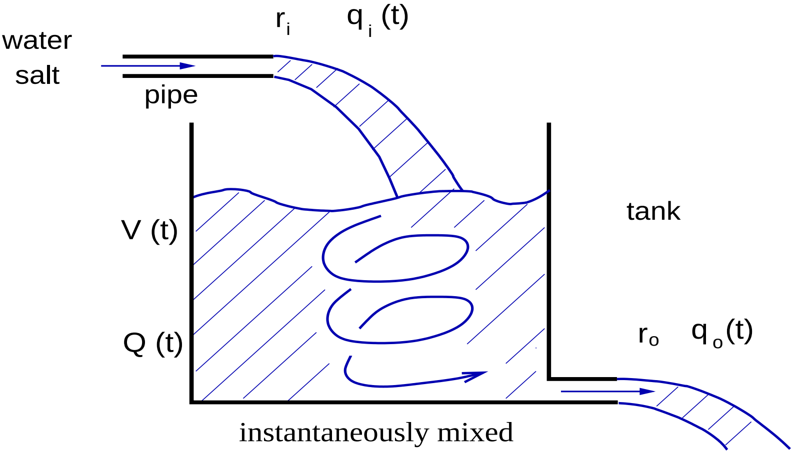
The experimental device.



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Analysis of the mathematical model.

Eqs. (4) and (1) imply

$$V(t) = (r_i - r_o)\, t + V_0, \tag{5}$$

where $V(0) = V_0$ is the initial volume of water in the tank.

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If r, q_i, Q_0 and V_0 are given, find $Q(t)$.

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Solution: Always holds $Q'(t) = a(t)\, Q(t) + b(t)$.
In this case:

$$a(t) = -\frac{r_o}{(r_i - r_o)\, t + V_0} \quad \Rightarrow \quad a(t) = -\frac{r}{V_0} = -a_0,$$

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Predictions for particular situations.

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Assume that $r_i = r_o = r$ and q_i are constants.
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We need to solve the IVP:

$$Q'(t) = -a_0\,Q(t) + b_0, \quad Q(0) = Q_0.$$

Predictions for particular situations.

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Assume that $r_i = r_o = r$ and q_i are constants.
If r, q_i, Q_0 and V_0 are given, find $Q(t)$.

Solution: Recall the IVP: $Q'(t) + a_0\, Q(t) = b_0, \quad Q(0) = Q_0$.

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Predictions for particular situations.

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Integrating factor method:

$$A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad e^{a_0 t} Q(t) = Q_0 + \int_0^t e^{a_0 s}\, b_0\, ds.$$

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$$Q(t) = e^{-a_0 t} \Big[Q_0 + \frac{b_0}{a_0}\big(e^{a_0 t} - 1\big)\Big].$$

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$$Q(t) = e^{-a_0 t} \Big[Q_0 + \frac{b_0}{a_0} \big(e^{a_0 t} - 1 \big) \Big]. = \Big(Q_0 - \frac{b_0}{a_0} \Big)\, e^{-a_0 t} + \frac{b_0}{a_0}.$$

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But $\dfrac{b_0}{a_0} = r q_i\, \dfrac{V_0}{r} = q_i V_0$, and $a_0 = \dfrac{r}{V_0}$. We conclude:

$$Q(t) = \left(Q_0 - q_i V_0\right) e^{-rt/V_0} + q_i V_0.$$

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Assume that $r_i = r_o = r$ and q_i are constants.
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Particular cases:

- $\dfrac{Q_0}{V_0} > q_i$;
- $\dfrac{Q_0}{V_0} = q_i$, so $Q(t) = Q_0$;
- $\dfrac{Q_0}{V_0} < q_i$.

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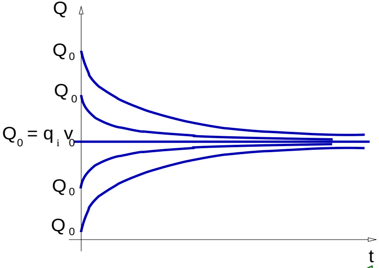

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Predictions for particular situations.

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Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example.

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If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0$,

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Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0$, we get

$$Q(t) = Q_0\, e^{-rt/V_0}.$$

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$$Q(t) = Q_0\, e^{-rt/V_0}.$$

Since $V(t) = (r_i - r_o)\, t + V_0$ and $r_i = r_o$, we obtain $V(t) = V_0$.

Predictions for particular situations.

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Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0$, we get

$$Q(t) = Q_0\, e^{-rt/V_0}.$$

Since $V(t) = (r_i - r_o)\, t + V_0$ and $r_i = r_o$, we obtain $V(t) = V_0$.
So $q(t) = Q(t)/V(t)$ is given by $q(t) = \dfrac{Q_0}{V_0}\, e^{-rt/V_0}$.

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Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0$, we get

$$Q(t) = Q_0\, e^{-rt/V_0}.$$

Since $V(t) = (r_i - r_o)\, t + V_0$ and $r_i = r_o$, we obtain $V(t) = V_0$.
So $q(t) = Q(t)/V(t)$ is given by $q(t) = \dfrac{Q_0}{V_0}\, e^{-rt/V_0}$. Therefore,

$$\frac{1}{100}\,\frac{Q_0}{V_0} = q(t_1)$$

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If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0$, we get

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So $q(t) = Q(t)/V(t)$ is given by $q(t) = \dfrac{Q_0}{V_0}\, e^{-rt/V_0}$. Therefore,

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So $q(t) = Q(t)/V(t)$ is given by $q(t) = \dfrac{Q_0}{V_0}\, e^{-rt/V_0}$. Therefore,

$$\frac{1}{100}\,\frac{Q_0}{V_0} = q(t_1) = \frac{Q_0}{V_0}\, e^{-rt_1/V_0} \quad \Rightarrow \quad e^{-rt_1/V_0} = \frac{1}{100}.$$

Predictions for particular situations.

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Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \dfrac{1}{100}$.

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Solution: Recall: $e^{-rt_1/V_0} = \dfrac{1}{100}$. Then,

$$-\frac{r}{V_0}\,t_1 = \ln\Big(\frac{1}{100}\Big)$$

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If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \dfrac{1}{100}$. Then,

$$-\frac{r}{V_0}\,t_1 = \ln\Big(\frac{1}{100}\Big) = -\ln(100)$$

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If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

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$$-\frac{r}{V_0}\,t_1 = \ln\Big(\frac{1}{100}\Big) = -\ln(100) \quad\Rightarrow\quad \frac{r}{V_0}\,t_1 = \ln(100).$$

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Solution: Recall: $e^{-rt_1/V_0} = \dfrac{1}{100}$. Then,

$$-\frac{r}{V_0}\,t_1 = \ln\Big(\frac{1}{100}\Big) = -\ln(100) \quad \Rightarrow \quad \frac{r}{V_0}\,t_1 = \ln(100).$$

We conclude that $t_1 = \dfrac{V_0}{r}\,\ln(100)$.

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If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \dfrac{1}{100}$. Then,

$$-\frac{r}{V_0}\,t_1 = \ln\Big(\frac{1}{100}\Big) = -\ln(100) \quad \Rightarrow \quad \frac{r}{V_0}\,t_1 = \ln(100).$$

We conclude that $t_1 = \dfrac{V_0}{r}\,\ln(100)$.

In this case: $t_1 = 100\,\ln(100)$. $\triangleleft$

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ are constants. If $r = 5x10^6$ gal/year, $q_i(t) = 2 + \sin(2t)$ grams/gal, $V_0 = 10^6$ gal, $Q_0 = 0$, find $Q(t)$.

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Assume that $r_i = r_o = r$ are constants. If $r = 5x10^6$ gal/year, $q_i(t) = 2 + \sin(2t)$ grams/gal, $V_0 = 10^6$ gal, $Q_0 = 0$, find $Q(t)$.

Solution: Recall: $Q'(t) = a(t)\, Q(t) + b(t)$.

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Solution: Recall: $Q'(t) = a(t)\,Q(t) + b(t)$. In this case:

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0}$$

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Solution: Recall: $Q'(t) = a(t)\,Q(t) + b(t)$. In this case:

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0} \quad \Rightarrow \quad a(t) = -\frac{r}{V_0} = -a_0,$$

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Solution: Recall: $Q'(t) = a(t)\,Q(t) + b(t)$. In this case:

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0} \quad \Rightarrow \quad a(t) = -\frac{r}{V_0} = -a_0,$$

$$b(t) = r_i\,q_i(t) \quad \Rightarrow \quad b(t) = r\big[2 + \sin(2t)\big].$$

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ are constants. If $r = 5x10^6$ gal/year, $q_i(t) = 2 + \sin(2t)$ grams/gal, $V_0 = 10^6$ gal, $Q_0 = 0$, find $Q(t)$.

Solution: Recall: $Q'(t) = a(t)\,Q(t) + b(t)$. In this case:

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0} \quad \Rightarrow \quad a(t) = -\frac{r}{V_0} = -a_0,$$

$$b(t) = r_i\,q_i(t) \quad \Rightarrow \quad b(t) = r\big[2 + \sin(2t)\big].$$

We need to solve the IVP: $Q'(t) = -a_0\,Q(t) + b(t),\; Q(0) = 0.$

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We conclude: $Q(t) = re^{-rt/V_0} \int_0^t e^{rs/V_0} \big[2 + \sin(2s)\big]\, ds.$

Exact equations (Sect. 1.4).

- Exact differential equations.
- The Poincaré Lemma.
- Implicit solutions and the potential function.
- Generalization: The integrating factor method.

Exact differential equations.

Definition

Given an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$ and continuously differentiable functions $M, N : R \to \mathbb{R}$,

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Recall: we use the notation: $\partial_t N = \dfrac{\partial N}{\partial t}$, and $\partial_u M = \dfrac{\partial M}{\partial u}$.

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Show whether the differential equation below is exact,

$$2ty(t)\, y'(t) + 2t + y^2(t) = 0.$$

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Remark: The ODE above is not separable and non-linear.

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Show whether the differential equation below is exact,

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This implies that $\partial_t N(t,u) \neq \partial_u M(t,u)$. $\triangleleft$

Exact equations (Sect. 1.4).

- Exact differential equations.
- **The Poincaré Lemma.**
- Implicit solutions and the potential function.
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Remark: The coefficients N and M of an exact equations are the derivatives of a potential function ψ.

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Lemma (Poincaré)

Given an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, *the continuously differentiable functions* $M, N : R \to \mathbb{R}$ *satisfy the equation*

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iff there exists a twice continuously differentiable function $\psi : R \to \mathbb{R}$, *called* potential function, *such that for all* $(t, u) \in R$ *holds*

$$\partial_u \psi(t, u) = N(t, u), \qquad \partial_t \psi(t, u) = M(t, u).$$

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($\Rightarrow$) Difficult: Poincaré, 1880.

The Poincaré Lemma.

Example

Show that the function $\psi(t,u) = t^2 + tu^2$ is the potential function for the exact differential equation

$$2ty(t)\,y'(t) + 2t + y^2(t) = 0.$$

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The Poincaré Lemma.

Example

Show that the function $\psi(t,u) = t^2 + tu^2$ is the potential function for the exact differential equation

$$2ty(t)\,y'(t) + 2t + y^2(t) = 0.$$

Solution: We already saw that the differential equation above is exact, since the functions M and N,

$$\left.\begin{aligned} N(t,u) &= 2tu, \\ M(t,u) &= 2t + u^2 \end{aligned}\right\} \quad \Rightarrow \quad \partial_t N = 2u = \partial_u M.$$

The potential function is $\psi(t,u) = t^2 + tu^2$, since

$$\partial_t \psi = 2t + u^2 = M, \qquad \partial_u \psi = 2tu = N. \qquad \lhd$$

Remark: The Poincaré Lemma only states necessary and sufficient conditions on N and M for the existence of ψ.

Exact equations (Sect. 1.4).

- Exact differential equations.
- The Poincaré Lemma.
- **Implicit solutions and the potential function.**
- Generalization: The integrating factor method.

Implicit solutions and the potential function.

Theorem (Exact differential equations)

Let $M, N : R \to \mathbb{R}$ be continuously differentiable functions on an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$. If the differential equation

$$N(t, y(t))\, y'(t) + M(t, y(t)) = 0 \tag{8}$$

is exact, then every solution $y : (t_1, t_2) \to \mathbb{R}$ must satisfy the algebraic equation

$$\psi(t, y(t)) = c,$$

where $c \in \mathbb{R}$ and $\psi : R \to \mathbb{R}$ is a potential function for Eq. (8).

Implicit solutions and the potential function.

Theorem (Exact differential equations)

Let $M, N : R \to \mathbb{R}$ *be continuously differentiable functions on an open rectangle* $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$*. If the differential equation*

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Proof: $0 = N(t, y)\, y' + M(t, y)$

Implicit solutions and the potential function.

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Proof: $0 = N(t, y)\, y' + M(t, y) = \partial_y \psi(t, y)\, \dfrac{dy}{dt} + \partial_t \psi(t, y)).$

Implicit solutions and the potential function.

Theorem (Exact differential equations)

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Implicit solutions and the potential function.

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□

Implicit solutions and the potential function.

Example

Find all solutions y to the equation

$$\left[\sin(t) + t^2 e^{y(t)} - 1\right] y'(t) + y(t)\cos(t) + 2te^{y(t)} = 0.$$

Implicit solutions and the potential function.

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$$\partial_u \psi(t,u) = N(t,u), \quad \partial_t \psi(t,u) = M(t,u).$$

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These are actually equations for ψ.

Implicit solutions and the potential function.

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$$\partial_u \psi(t,u) = N(t,u), \quad \partial_t \psi(t,u) = M(t,u).$$

These are actually equations for ψ. From the first one,

$$\psi(t,u) = \int \big[\sin(t) + t^2 e^u - 1\big]\, du + g(t).$$

Implicit solutions and the potential function.

Example

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Solution: $\psi(t,u) = \int \left[\sin(t) + t^2 e^{u} - 1\right] du + g(t).$

Implicit solutions and the potential function.

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Introduce this expression into $\partial_t \psi(t,u) = M(t,u)$,

Implicit solutions and the potential function.

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Implicit solutions and the potential function.

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Therefore, $g'(t) = 0$, so we choose $g(t) = 0$.

Implicit solutions and the potential function.

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Therefore, $g'(t) = 0$, so we choose $g(t) = 0$. We obtain,

$$\psi(t,u) = u\sin(t) + t^2 e^u - u.$$

So the solution y satisfies $y(t)\sin(t) + t^2 e^{y(t)} - y(t) = c.$ $\triangleleft$

Exact equations (Sect. 1.4).

- Exact differential equations.
- The Poincaré Lemma.
- Implicit solutions and the potential function.
- **Generalization: The integrating factor method.**

Remark:
Sometimes a non-exact equation can we transformed into an exact equation multiplying the equation by an integrating factor. Just like in the case of linear differential equations.

Generalization: The integrating factor method.

Theorem (Integrating factor)

Let $M, N : R \to \mathbb{R}$ *be continuously differentiable functions on* $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, *with* $N \neq 0$. *If the equation*

$$N(t, y(t))\, y'(t) + M(t, y(t)) = 0$$

is not exact, that is, $\partial_t N(t, u) \neq \partial_u M(t, u)$,

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$$\frac{1}{N(t, u)} \big[\partial_u M(t, u) - \partial_t N(t, u)\big]$$

does not depend on the variable u,

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is not exact, that is, $\partial_t N(t, u) \neq \partial_u M(t, u)$, *and if the function*

$$\frac{1}{N(t, u)} \big[\partial_u M(t, u) - \partial_t N(t, u)\big]$$

does not depend on the variable u, *then the equation*

$$\mu(t) \big[N(t, y(t))\, y'(t) + M(t, y(t))\big] = 0$$

is exact, where $\dfrac{\mu'(t)}{\mu(t)} = \dfrac{1}{N(t, u)} \big[\partial_u M(t, u) - \partial_t N(t, u)\big].$

Generalization: The integrating factor method.

Example

Find all solutions y to the differential equation

$$\big[t^2 + t\,y(t)\big]\,y'(t) + \big[3t\,y(t) + y^2(t)\big] = 0.$$

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Find all solutions y to the differential equation

$$\big[t^2 + t\,y(t)\big]\,y'(t) + \big[3t\,y(t) + y^2(t)\big] = 0.$$

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$$\frac{\big[\partial_u M(t,u) - \partial_t N(t,u)\big]}{N(t,u)}$$

Generalization: The integrating factor method.

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$$\frac{\big[\partial_u M(t,u) - \partial_t N(t,u)\big]}{N(t,u)} = \frac{1}{(t^2 + tu)}\big[(3t + 2u) - (2t + u)\big]$$

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$$\frac{\left[\partial_u M(t,u) - \partial_t N(t,u)\right]}{N(t,u)} = \frac{1}{t(t + u)}\,(t + u)$$

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$$\frac{\left[\partial_u M(t,u) - \partial_t N(t,u)\right]}{N(t,u)} = \frac{1}{t(t+u)}\,(t+u) = \frac{1}{t}.$$

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Find all solutions y to the differential equation

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We find a function μ solution of $\dfrac{\mu'}{\mu} = \dfrac{\big[\partial_u M - \partial_t N\big]}{N},$

Generalization: The integrating factor method.

Example

Find all solutions y to the differential equation

$$\left[t^2 + t\,y(t)\right] y'(t) + \left[3t\,y(t) + y^2(t)\right] = 0.$$

Solution: $\dfrac{\left[\partial_u M(t,u) - \partial_t N(t,u)\right]}{N(t,u)} = \dfrac{1}{t}.$

We find a function μ solution of $\dfrac{\mu'}{\mu} = \dfrac{\left[\partial_u M - \partial_t N\right]}{N}$, that is

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Generalization: The integrating factor method.

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Therefore, the equation below is exact:

$$\big[t^3 + t^2\,y(t)\big]\,y'(t) + \big[3t^2\,y(t) + t\,y^2(t)\big] = 0.$$

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$$\partial_u \psi(t,u) = \tilde{N}(t,u), \qquad \partial_t \psi(t,u) = \tilde{M}(t,u).$$

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So $g'(t) = 0$ and we choose $g(t) = 0$.

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And every solution y satisfies $t^3\,y(t) + \dfrac{1}{2}\,t^2\big[y(t)\big]^2 = c.$ $\triangleleft$