The integrating factor method (Sect. 1.1)

- Overview of differential equations.
- Linear Ordinary Differential Equations.
- The integrating factor method.
  - Constant coefficients.
  - The Initial Value Problem.

Overview of differential equations.

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$$\frac{d^2}{dt^2}\mathbf{x}(t) = \frac{1}{m}\,\mathbf{F}(t, \mathbf{x}(t)),$$

with m the particle mass and $\mathbf{F}$ the force acting on the particle.

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The wave equation is a PDE: The unknown is $u(t, x)$, a function that depends on two variables, and the equation is

$$\frac{\partial^2}{\partial t^2}u(t, x) = v^2\frac{\partial^2}{\partial x^2}u(t, x),$$

with v the wave speed. Sound propagation in air is described by a wave equation, where u represents the air pressure.

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Given constants $a, b \in \mathbb{R}$ with $a \neq 0$, the linear differential equation

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has infinitely many solutions, one for each value of $c \in \mathbb{R}$, given by

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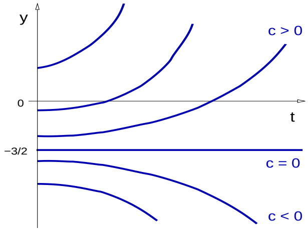

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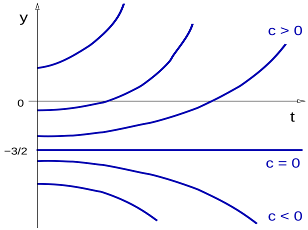

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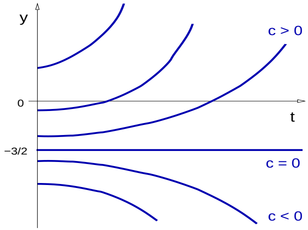

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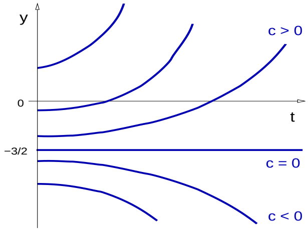

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  - Constant coefficients.
  - **The Initial Value Problem.**

The Initial Value Problem.

Definition

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Theorem (Constant coefficients)

Given constants $a, b, t_0, y_0 \in \mathbb{R}$, with $a \neq 0$, the initial value problem

$$y' = a\, y + b, \qquad y(t_0) = y_0$$

has the unique solution

$$y(t) = \Big(y_0 + \frac{b}{a}\Big) e^{a(t-t_0)} - \frac{b}{a}.$$

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Find the solution to the initial value problem

$$y' = 2y + 3, \qquad y(0) = 1.$$

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We conclude that $y(t) = \frac{2}{3}\,e^{-3t} + \frac{1}{3}$. ◁

Linear Variable coefficient equations (Sect. 2.1)

- Review: Linear constant coefficient equations.
- The Initial Value Problem.
- Linear variable coefficients equations.
- The Bernoulli equation: A nonlinear equation.

Review: Linear constant coefficient equations

Definition

Given functions $a, b : \mathbb{R} \to \mathbb{R}$, a *first order linear ODE* in the unknown function $y : \mathbb{R} \to \mathbb{R}$ is the equation

$$y'(t) = a(t)\,y(t) + b(t).$$

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(a) A constant coefficients first order linear ODE is given by

$$y'(t) = -2\, y(t) + 3.$$

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Here $a(t) = -2/t$ and $b(t) = 4t$.

Review: Linear constant coefficient equations

Theorem (Constant coefficients)

Given constants $a, b \in \mathbb{R}$ with $a \neq 0$, the linear differential equation

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has infinitely many solutions, one for each value of $c \in \mathbb{R}$, given by

$$y(t) = c\, e^{at} - \frac{b}{a}.$$

Review: Linear constant coefficient equations

Theorem (Constant coefficients)

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Remarks:

(a) A proof is given in the Lecture Notes.

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Remarks:

(a) A proof is given in the Lecture Notes.

(b) Solutions to first order linear ODE can be obtained using the integrating factor method.

Review: Linear constant coefficient equations

Example

Find all functions y solution of the ODE $y' = 2y + 3$.

Review: Linear constant coefficient equations

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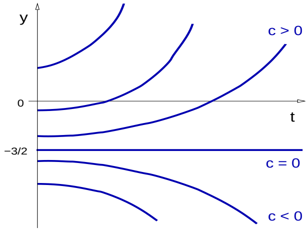

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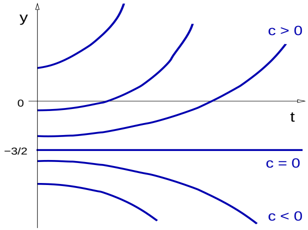

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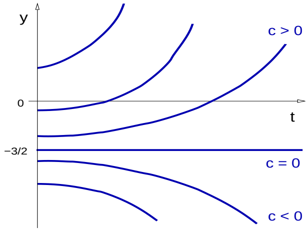

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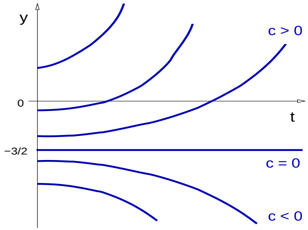

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Verification: $y' = 2c\, e^{2t}$, but we know that $2c\, e^{2t} = 2y + 3$, therefore we conclude that y satisfies the ODE $y' = 2y + 3$. $\triangleleft$

Linear Variable coefficient equations (Sect. 2.1)

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- **The Initial Value Problem.**
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- The Bernoulli equation: A nonlinear equation.

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The Initial Value Problem

Definition

The *Initial Value Problem* (IVP) for a linear ODE is the following: Given functions $a, b : \mathbb{R} \to \mathbb{R}$ and constants t_0, $y_0 \in R$, find a solution $y : \mathbb{R} \to \mathbb{R}$ of the problem

$$y' = a(t)\, y + b(t), \qquad y(t_0) = y_0.$$

Remark: The initial condition selects one solution of the ODE.

Theorem (Constant coefficients)

Given constants $a, b, t_0, y_0 \in \mathbb{R}$, with $a \neq 0$, the initial value problem

$$y' = a\, y + b, \qquad y(t_0) = y_0$$

has the unique solution

$$y(t) = \Big(y_0 + \frac{b}{a}\Big) e^{a(t-t_0)} - \frac{b}{a}.$$

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We conclude that $y(t) = \frac{5}{2}\, e^{2t} - \frac{3}{2}$. $\triangleleft$

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Linear Variable coefficient equations (Sect. 2.1)

- Review: Linear constant coefficient equations.
- The Initial Value Problem.
- **Linear variable coefficients equations.**
- The Bernoulli equation: A nonlinear equation.

Linear variable coefficients equations

Theorem (Variable coefficients)

Given continuous functions $a, b : \mathbb{R} \to \mathbb{R}$ *and given constants* $t_0, y_0 \in \mathbb{R}$, *the IVP*

$$y' = a(t)y + b(t) \qquad y(t_0) = y_0$$

has the unique solution

$$y(t) = e^{A(t)} \left[y_0 + \int_{t_0}^{t} e^{-A(s)}\, b(s)ds \right],$$

where we have introduced the function $A(t) = \displaystyle\int_{t_0}^{t} a(s)ds.$

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Remarks:

(a) The function $\mu(t) = e^{-A(t)}$ is called an integrating factor.

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(b) See the proof in the Lecture Notes.

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Example

Find the solution y to the IVP

$$t\,y' = -2y + 4t^2, \qquad y(1) = 2.$$

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$$f(t) = \int_1^t \frac{2}{s}\,ds = 2\big[\ln(t) - \ln(1)\big]$$

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Therefore, $\mu(t) = e^{f(t)} = t^2$.

Linear variable coefficients equations

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We conclude that $y(t) = t^2 + \dfrac{1}{t^2}$. $\triangleleft$

Linear Variable coefficient equations (Sect. 2.1)

- Review: Linear constant coefficient equations.
- The Initial Value Problem.
- Linear variable coefficients equations.
- **The Bernoulli equation: A nonlinear equation.**

The Bernoulli equation

Remark: The Bernoulli equation is a non-linear differential equation that can be transformed into a linear differential equation.

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Definition

Given functions p, $q : \mathbb{R} \to \mathbb{R}$ and a real number n, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y' + p(t)\, y = q(t)\, y^n$$

is called the *Bernoulli equation*.

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Theorem

The function $y : \mathbb{R} \to \mathbb{R}$ is a solution of the Bernoulli equation for

$$y' + p(t)\, y = q(t)\, y^n, \qquad n \neq 1,$$

iff the function $v = 1/y^{(n-1)}$ is solution of the linear differential equation

$$v' - (n-1)p(t)\, v = -(n-1)q(t).$$

The Bernoulli equation

Example

Given arbitrary constants $a \neq 0$ and b, find every solution of the differential equation

$$y' = a\,y + b\,y^3.$$

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Introduce the function $v = 1/y^2$,

The Bernoulli equation

Example

Given arbitrary constants $a \neq 0$ and b, find every solution of the differential equation

$$y' = a\,y + b\,y^3.$$

Solution: This is a Bernoulli equation. Divide the equation by y^3,

$$\frac{y'}{y^3} = \frac{a}{y^2} + b.$$

Introduce the function $v = 1/y^2$, with derivative $v' = -2\Big(\frac{y'}{y^3}\Big)$,

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$$\frac{1}{y^2} = c\,e^{-2at} - \frac{b}{a} \quad \Rightarrow \quad y(t) = \pm\frac{1}{\bigl(c\,e^{-2at} - \frac{b}{a}\bigr)^{1/2}}. \qquad \triangleleft$$

Separable differential equations (Sect. 1.3).

- Separable ODE.
- Solutions to separable ODE.
- Explicit and implicit solutions.
- Homogeneous equations.

Separable ODE.

Definition

Given functions $h, g : \mathbb{R} \to \mathbb{R}$, a first order ODE on the unknown function $y : \mathbb{R} \to \mathbb{R}$ is called *separable* iff the ODE has the form

$$h(y)\, y'(t) = g(t).$$

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- The linear differential equation $y'(t) = -a(t)\,y(t) + b(t)$, with $b(t)$ non-constant, is not separable.

Separable differential equations (Sect. 1.3).

- Separable ODE.
- **Solutions to separable ODE.**
- Explicit and implicit solutions.
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Solutions to separable ODE.

Theorem (Separable equations)

If the functions $g, h : \mathbb{R} \to \mathbb{R}$ *are continuous, with* $h \neq 0$ *and with primitives* G *and* H*, respectively; that is,*

$$G'(t) = g(t), \qquad H'(u) = h(u),$$

then, the separable ODE

$$h(y)\, y' = g(t)$$

has infinitely many solutions $y : \mathbb{R} \to \mathbb{R}$ *satisfying the algebraic equation*

$$H(y(t)) = G(t) + c,$$

where $c \in \mathbb{R}$ *is arbitrary.*

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Remark: Given functions g, h, find their primitives G, H.

Solutions to separable ODE.

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Find all solutions y to the equation $y'(t) = \dfrac{t^2}{1 - y^2(t)}$.

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Hence we recover the Theorem expression: $H(y(t)) = G(t) + c$.

Solutions to separable ODE.

Remarks:

- The equation $y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c$ is algebraic in y, since there is no y' in the equation.

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Hence we recover the Theorem expression: $H(y(t)) = G(t) + c$.

Separable differential equations (Sect. 1.3).

- Separable ODE.
- Solutions to separable ODE.
- **Explicit and implicit solutions.**
- Homogeneous equations.

Explicit and implicit solutions.

Definition

Assume the notation in the Theorem above. The solution y of a separable ODE is given in *implicit form* iff function y is given by

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(a) $y(t) - \dfrac{y^3(t)}{3} = \dfrac{t^3}{3} + c$

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Homogeneous equations.

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The first order ODE $y'(t) = f\big(t, y(t)\big)$ is called *homogeneous* iff for every numbers $c, t, u \in \mathbb{R}$ the function f satisfies

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- So, there exists $F : \mathbb{R} \to \mathbb{R}$ such that $f(t, u) = F\Big(\frac{u}{t}\Big)$.
- Therefore, a first order ODE is homogeneous iff it has the form

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Show that the equation below is homogeneous,

$$(t - y)\, y' - 2y + 3t + \frac{y^2}{t} = 0.$$

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$$f(t, y) = \frac{2y - 3t - (y^2/t)}{t - y},$$

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and $f(t,y) = F(y/t)$. ◁

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We conclude that the differential equation is not homogeneous. $\triangleleft$

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Theorem

If the differential equation $y'(t) = f\big(t, y(t)\big)$ *is homogeneous, then the differential equation for the unknown* $v(t) = \dfrac{y(t)}{t}$ *is separable.*

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This last equation is separable. □

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Therefore, we introduce the change of unknown $v = y/t$,

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Therefore, we introduce the change of unknown $v = y/t$, so $y = t\,v$ and $y' = v + t\,v'$.

Homogeneous equations.

Example

Find all solutions y of the equation $y' = \dfrac{t^2 + 3y^2}{2ty}$.

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$$v + t\,v' = \frac{1 + 3v^2}{2v} \quad \Rightarrow \quad t\,v' = \frac{1 + 3v^2}{2v} - v$$

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We obtain the separable equation $v' = \dfrac{1}{t}\left(\dfrac{1 + v^2}{2v}\right)$.

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Example

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Solution: Recall: $v' = \dfrac{1}{t}\left(\dfrac{1+v^2}{2v}\right)$.

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Find all solutions y of the equation $y' = \dfrac{t^2 + 3y^2}{2ty}$.

Solution: Recall: $v' = \dfrac{1}{t}\left(\dfrac{1+v^2}{2v}\right)$. We rewrite and integrate it,

$$\frac{2v}{1+v^2}\,v' = \frac{1}{t}$$

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The substitution $u = 1 + v^2(t)$ implies $du = 2v(t)\, v'(t)\, dt$,

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But $u = e^{\ln(t)} e^{c_0}$,

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But $u = e^{\ln(t)} e^{c_0}$, so denoting $c_1 = e^{c_0}$, then $u = c_1 t$. Hence

$$1 + v^2 = c_1 t \quad \Rightarrow \quad 1 + \left(\frac{y}{t}\right)^2 = c_1 t \quad \Rightarrow \quad y(t) = \pm t\sqrt{c_1 t - 1}.$$