Solving the Heat Equation (Sect. 6.3).

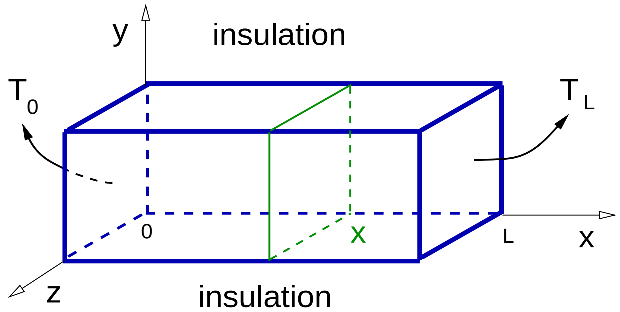
- Review: The Stationary Heat Equation.
- The Heat Equation.
- The Initial-Boundary Value Problem.
- The separation of variables method.
- An example of separation of variables.

Review: The Stationary Heat Equation.

Review: The Stationary Heat Equation describes the temperature distribution in a solid material in thermal equilibrium. The temperature is time-independent.

Problem: The time-independent temperature, T, of a bar of length L with insulated horizontal sides and vertical extremes kept at fixed temperatures T_0, T_L, is the solution of the BVP:

$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$



Remark: The heat transfer occurs only along the x-axis.

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The Heat Equation.

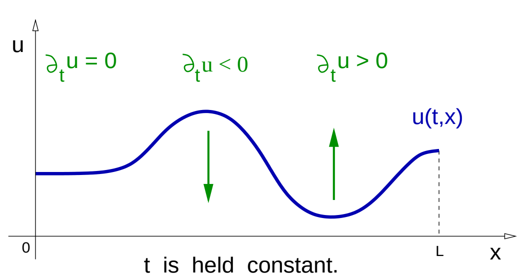
Remarks:

- The unknown of the problem is $u(t,x)$, the temperature of the bar at the time t and position x.
- The temperature does not depend on y or z.
- The one-dimensional Heat Equation is:

$$\partial_t u(t,x) = k\,\partial_x^2 u(t,x),$$

where $k > 0$ is the heat conductivity, units: $[k] = \dfrac{(\text{distance})^2}{(\text{time})}$.

- The Heat Equation is a Partial Differential Equation, PDE.



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The Initial-Boundary Value Problem.

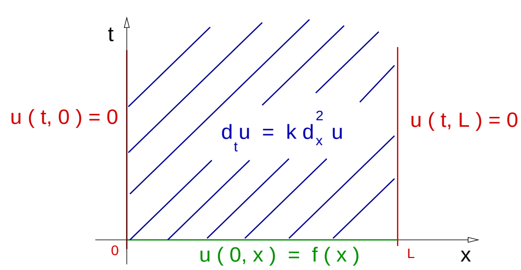
Definition

The IBVP for the one-dimensional Heat Equation is the following: Given a constant $k > 0$ and a function $f : [0, L] \to \mathbb{R}$ with $f(0) = f(L) = 0$, find $u : [0, \infty) \times [0, L] \to \mathbb{R}$ solution of

$$\partial_t u(t, x) = k\, \partial_x^2 u(t, x),$$

$$\text{I.C.:} \quad u(0, x) = f(x),$$

$$\text{B.C.:} \quad u(t, 0) = 0, \quad u(t, L) = 0.$$



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The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x).$$

where

- v_n: Solution of an IVP.
- w_n: Solution of a BVP, an eigenvalue-eigenfunction problem.
- c_n: Fourier Series coefficients.

Remark:
The separation of variables method does not work for every PDE.

The separation of variables method.

Summary:

- The idea is to transform the PDE into infinitely many ODEs.
- We describe this method in 6 steps.

Step 1:

One looks for solutions u given by an infinite series of simpler functions, u_n, that is,

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, u_n(t,x),$$

where u_n is simpler than u is the sense,

$$u_n(t,x) = v_n(t) \, w_n(x).$$

Here c_n are constants, $n = 1, 2, \cdots$.

The separation of variables method.

Step 2:

Introduce the series expansion for u into the Heat Equation,

$$\partial_t u - k \, \partial_x^2 u = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} c_n \left[\partial_t u_n - k \, \partial_x^2 u_n \right] = 0.$$

A sufficient condition for the equation above is: To find u_n, for $n = 1, 2, \cdots$, solutions of

$$\partial_t u_n - k \, \partial_x^2 u_n = 0.$$

Step 3:

Find $u_n(t,x) = v_n(t) \, w_n(x)$ solution of the IBVP

$$\partial_t u_n - k \, \partial_x^2 u_n = 0.$$

$$\text{I.C.:} \quad u_n(0,x) = w_n(x),$$

$$\text{B.C.:} \quad u_n(t,0) = 0, \quad u_n(t,L) = 0.$$

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

Notice:

$$\partial_t u_n(t,x) = \partial_t\big[v_n(t)\,w_n(x)\big] = w_n(x)\,\frac{dv_n}{dt}(t).$$

$$\partial_x^2 u_n(t,x) = \partial_x^2\big[v_n(t)\,w_n(x)\big] = v_n(t)\,\frac{d^2 w_n}{dx^2}(x).$$

Therefore, the equation $\partial_t u_n = k\partial_x^2 u_n$ is given by

$$w_n(x)\,\frac{dv_n}{dt}(t) = k\,v_n(t)\,\frac{d^2 w_n}{dx^2}(x)$$

$$\frac{1}{k\,v_n(t)}\,\frac{dv_n}{dt}(t) = \frac{1}{w_n(x)}\,\frac{d^2 w_n}{dx^2}(x).$$

Depends only on t = Depends only on x.

The separation of variables method.

Recall:

$$\frac{1}{k\,v_n(t)}\,\frac{dv_n}{dt}(t) = \frac{1}{w_n(x)}\,\frac{d^2 w_n}{dx^2}(x).$$

Depends only on t = Depends only on x.

- The Heat Equation has the following property:
  The left-hand side depends only on t, while the right-hand side depends only on x.
- When this happens in a PDE, one can use the separation of variables method on that PDE.
- We conclude that for appropriate constants λ_m holds

$$\frac{1}{k\,v_n(t)}\,\frac{dv_n}{dt}(t) = -\lambda_n, \qquad \frac{1}{w_n(x)}\,\frac{d^2 w_n}{dx^2}(x) = -\lambda_n.$$

- We have transformed the original PDE into infinitely many ODEs parametrized by n, positive integer.

The separation of variables method.

Summary Step 4: The original *IBVP* for the Heat Equation, PDE, can transformed into:

(a) We choose to solve the following IVP for v_n,

$$\frac{1}{k\, v_n(t)}\, \frac{dv_n}{dt}(t) = -\lambda_n, \quad \text{I.C.:} \quad v_n(0) = 1.$$

Remark: This choice of I.C. simplifies the problem.

(b) The BVP for w_n,

$$\frac{1}{w_n(x)}\, \frac{d^2 w_n}{dx^2}(x) = -\lambda_n, \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0.$$

Step 5:

(a) Solve the IVP for v_n.

(b) Solve the BVP for w_n.

The separation of variables method.

Step 5(a): Solving the IVP for v_n.

$$v_n'(t) + k\lambda_n\, v_n(t) = 0, \quad \text{I.C.:} \quad v_n(0) = 1.$$

The integrating factor method implies that $\mu(t) = e^{k\lambda_n t}$.

$$e^{k\lambda_n t} v_n'(t) + k\lambda_n\, e^{k\lambda_n t}\, v_n(t) = 0 \quad \Rightarrow \quad \left[e^{k\lambda_n t} v_n(t)\right]' = 0.$$

$$e^{k\lambda_n t} v_n(t) = c_n \quad \Rightarrow \quad v_n(t) = c_n\, e^{-k\lambda_n t}.$$

$$1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-k\lambda_n t}.$$

The separation of variables method.

Step 5(a): Recall: $v_n(t) = e^{-k\lambda_n t}$.

Step 5(b): Eigenvalue-eigenvector problem for w_n:
Find the eigenvalues λ_n and the non-zero eigenfunctions w_n solutions of the BVP

$$w_n''(x) + \lambda_n\, w_n(x) = 0 \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0.$$

We know that this problem has solution only for $\lambda_n > 0$.
Denote: $\lambda_n = \mu_n^2$. Proposing $w_n(x) = e^{r_n x}$, we get that

$$p(r_n) = r_n^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm \mu_n i$$

The real-valued general solution is

$$w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x).$$

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0) = c_1 \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(L) = c_2 \sin(\mu_n L), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n L) = 0.$$

$$\mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L} \quad \Rightarrow \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

Choosing $c_2 = 1$, we get $w_n(x) = \sin\left(\frac{n\pi x}{L}\right)$.

We conclude that: $u_n(t,x) = e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, \cdots$.

The separation of variables method.

Step 6: Recall: $u_n(t,x) = e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big)$.

Compute the solution to the IBVP for the Heat Equation,

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, u_n(t,x).$$

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big).$$

By construction, this solution satisfies the boundary conditions,

$$u(t,0) = 0, \qquad u(t,L) = 0.$$

Given a function f with $f(0) = f(L) = 0$, the solution u above satisfies the initial condition $f(x) = u(0,x)$ iff holds

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi x}{L}\Big).$$

The separation of variables method.

Recall:

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big), \quad f(x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi x}{L}\Big).$$

This is a Sine Series for f. The coefficients c_n are computed in the usual way. Recall the orthogonality relation

$$\int_0^L \sin\Big(\frac{n\pi x}{L}\Big) \sin\Big(\frac{m\pi x}{L}\Big)\, dx = \begin{cases} 0, & m \neq n, \\ \frac{L}{2}, & m = n. \end{cases}$$

Multiply the equation for u by $\sin\Big(\frac{m\pi x}{L}\Big)$ nd integrate,

$$\sum_{n=1}^{\infty} c_n \int_0^L \sin\Big(\frac{n\pi x}{L}\Big) \sin\Big(\frac{m\pi x}{L}\Big)\, dx = \int_0^L f(x) \sin\Big(\frac{m\pi x}{L}\Big)\, dx.$$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \quad u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big).$$

The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, v_n(t)\, w_n(x).$$

where

- v_n: Solution of an IVP.
- w_n: Solution of a BVP, an eigenvalue-eigenfunction problem.
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Remark:
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An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Let $u_n(t,x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv}{dt}(t) = v_n(t)\,\frac{d^2w}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n'(t) + \frac{\lambda_n}{4}\, v_n(t) = 0, \qquad w_n''(x) + \lambda_n\, w_n(x) = 0.$$

We solve for v_n with the initial condition $v_n(0) = 1$.

$$e^{\frac{\lambda_n}{4}t}\, v_n'(t) + \frac{\lambda_n}{4}\, e^{\frac{\lambda_n}{4}t}\, v_n(t) = 0 \quad \Rightarrow \quad \left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0.$$

An example of separation of variables.

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Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(L) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply

$$0 = w_n(0) = c_1, \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n\pi}{2}\right)^2, \qquad w_n(x) = \sin\left(\frac{n\pi x}{2}\right).$$

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-(\frac{n\pi}{4})^2 t} \sin\left(\frac{n\pi x}{2}\right).$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $u(t,x) = \displaystyle\sum_{n=1}^{\infty} c_n\, e^{-(\frac{n\pi}{4})^2 t} \sin\left(\frac{n\pi x}{2}\right).$

The initial condition is $3 \sin\left(\dfrac{\pi x}{2}\right) = \displaystyle\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right).$

The orthogonality of the sine functions implies

$$3\int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx.$$

If $m \neq 1$, then $0 = c_m \frac{2}{2}$, that is, $c_m = 0$ for $m \neq 1$. Therefore,

$$3\sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right) \quad \Rightarrow \quad c_1 = 3.$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: We conclude that

$$u(t,x) = 3\, e^{-(\frac{\pi}{4})^2 t}\, \sin\Big(\frac{\pi x}{2}\Big).$$