## Review Exam 4.

- Sections 5.1-5.9, 6.1-6.2.
- 7 problems.
- 60 minutes.
- 70 attempts.
  - Review of Linear Algebra (5.2,5.3, 5.5).
  - Basic Theory of first order systems (5.1, 5.4, 5.6).
  - Homogeneous constant coefficients systems:
    - Real and different eigenvalues (5.7).
    - Complex eigenvalues (5.8).
    - Real and repeated eigenvalues (5.9).
  - Eigenvalue-Eigenfunction, boundary value probl. (6.1).
  - Fourier series and even/odd extensions (6.2).


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## Homogeneous constant coefficients systems

Example

Find the real-valued general solution of

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \qquad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 2 \\ -2 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 + 4 = 0$$

$$(\lambda - 1)^2 = -4 \quad \Rightarrow \quad \lambda_{\pm} = 1 \pm 2i.$$

Eigenvector for $\lambda_+$.

$$(A - \lambda_+ I) = \begin{bmatrix} 1-(1+2i) & 2 \\ -2 & 1-(1+2i) \end{bmatrix} = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}.$$

## Homogeneous constant coefficients systems

Example

Find the real-valued general solution of

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \qquad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = 1 \pm 2i$, $(A - \lambda_+ I) = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}$.

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \to \begin{bmatrix} 2 & 2i \\ -2 & -2i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = -iv_2.$$

Choosing $v_2 = 1$, we get $v_1 = -i$, that is,

$$\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}^{(-)} = \overline{\mathbf{v}}^{(+)} = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

## Homogeneous constant coefficients systems

Example

Find the real-valued general sol. $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

Solution: Recall: $\lambda_{\pm} = 1 \pm 2i$, and $\mathbf{v}^{(\pm)} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$.

Also recalling: If $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b} i$, then

$$\mathbf{x}^{(1)}(t) = e^{\alpha t}\big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big],$$

$$\mathbf{x}^{(2)}(t) = e^{\alpha t}\big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big].$$

$$\mathbf{x}^{(1)} = e^{t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(2t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(2t)\right) \Rightarrow \mathbf{x}^{(1)} = e^{t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}.$$

$$\mathbf{x}^{(2)} = e^{t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(2t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(2t)\right) \Rightarrow \mathbf{x}^{(2)} = e^{t} \begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}.$$

## Hint to remember formulas for $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

Remark: The formulas for

$$\mathbf{x}^{(1)}(t) = e^{\alpha t}\big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big],$$

$$\mathbf{x}^{(2)}(t) = e^{\alpha t}\big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big].$$

are the real and imaginary part of $\tilde{\mathbf{x}}^{(+)} = (\mathbf{a} + \mathbf{b}i)\, e^{(\alpha+\beta i)t}$. Indeed,

$$\tilde{\mathbf{x}}^{(+)} = (\mathbf{a} + \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big]\, e^{\alpha t}.$$

$$\tilde{\mathbf{x}}^{(+)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\sin(\beta t)\big]\, e^{\alpha t} + i\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big]\, e^{\alpha t}.$$

## Homogeneous constant coefficients systems

### Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & \sqrt{2} \\ \sqrt{2} & (-2-\lambda) \end{vmatrix} = (\lambda+2)(\lambda+3) - 2 = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2}\left[-5 \pm \sqrt{25-16}\right] = \frac{1}{2}\left[-5 \pm 3\right]$$

Hence $\lambda_+ = -1$, $\lambda_- = -4$. Eigenvector for $\lambda_+$.

$$(A + I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \to \begin{bmatrix} 2 & -\sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 2 & -\sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

$2v_1 = \sqrt{2}\,v_2$. Choosing $v_1 = 1$ and $v_2 = \sqrt{2}$, we get $\mathbf{v}^{(+)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$.

## Homogeneous constant coefficients systems

### Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = -1$, $\lambda_- = -4$, and $\mathbf{v}^{(+)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$.
Eigenvector for $\lambda_-$.

$$(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \to \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

$v_1 = -\sqrt{2}\,v_2$. Choosing $v_1 = 1$ and $v_2 = -\dfrac{1}{\sqrt{2}}$, so, $\mathbf{v}^{(-)} = \begin{bmatrix} 1 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$.

Fundamental solutions: $\mathbf{x}^{(+)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t}$, $\mathbf{x}^{(-)} = \begin{bmatrix} 1 \\ \frac{-1}{\sqrt{2}} \end{bmatrix} e^{-4t}$.

General solution: $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ \frac{-1}{\sqrt{2}} \end{bmatrix} e^{-4t}$. ◁

## Homogeneous constant coefficients systems

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & 4 \\ -1 & (1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4-4}\right] = -1.$$

Hence $\lambda_+ = \lambda_- = -1$. Eigenvector for $\lambda_\pm$.

$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

$v_1 = 2\,v_2$. Choosing $v_1 = 2$ and $v_2 = 1$, we get $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

## Homogeneous constant coefficients systems

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_\pm = -1$, and $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Find $\mathbf{w}$ solution of $(A + I)\mathbf{w} = \mathbf{v}$.

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $w_1 = 2w_2 - 1$, that is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Choose $w_2 = 0$, so $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

## Homogeneous constant coefficients systems

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = -1$, $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Fundamental sol: $\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$, $\mathbf{x}^{(2)} = \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

General sol: $\mathbf{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

## Homogeneous constant coefficients systems

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\mathbf{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

Initial condition: $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$,

that is, $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, also, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

The solution is $\mathbf{x} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + 5 \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$. $\lhd$

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## Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that $r$ is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$\mu = \frac{n\pi}{8}, \quad \lambda = \Big(\frac{n\pi}{8}\Big)^2, \quad y_n(x) = \sin\Big(\frac{n\pi x}{8}\Big), \quad n = 1, 2, \cdots \triangleleft$$

## Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

$$8\mu = (2n+1)\frac{\pi}{2}, \quad \Rightarrow \quad \mu = \frac{(2n+1)\pi}{16}.$$

Then, for $n = 1, 2, \cdots$ holds

$$\lambda = \Big[\frac{(2n+1)\pi}{16}\Big]^2, \quad y_n(x) = \sin\Big(\frac{(2n+1)\pi x}{16}\Big). \qquad \triangleleft$$

## Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1 \mu \sin(\mu 8), \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$8\mu = n\pi, \quad \Rightarrow \quad \mu = \frac{n\pi}{8}.$$

Then, choosing $c_1 = 1$, for $n = 1, 2, \cdots$ holds

$$\lambda = \Big(\frac{n\pi}{8}\Big)^2, \quad y_n(x) = \cos\Big(\frac{n\pi x}{8}\Big).$$

## Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

$$0 = y'(8) = 0.$$

Then, choosing $c_1 = 1$, holds,

$$\lambda = 0, \qquad y_0(x) = 1. \qquad \triangleleft$$

## Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that $r$ is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \;\Rightarrow\; c_1 = -\frac{\sin(\pi/3)}{\cos(\pi/3)}.$$

$$c_1 = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3} \;\Rightarrow\; y(x) = -\sqrt{3}\cos(x) + \sin(x). \quad \triangleleft$$

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## Fourier series and even/odd extensions

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[ a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since $f$ is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

$$b_n = 2 \int_0^1 (-1) \sin(n\pi x)\, dx = (-2)\, \frac{(-1)}{n\pi} \cos(n\pi x)\Big|_0^1,$$

$$b_n = \frac{2}{n\pi} \big[\cos(n\pi) - 1\big] \quad \Rightarrow \quad b_n = \frac{2}{n\pi} \big[(-1)^n - 1\big].$$

## Fourier series and even/odd extensions

### Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \frac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \frac{2}{2k\pi}\big[(-1)^{2k} - 1\big] = 0$.

If $n = 2k-1$,
$$b_{(2k-1)} = \frac{2}{(2k-1)\pi}\big[(-1)^{2k-1} - 1\big] = -\frac{4}{(2k-1)\pi}.$$

We conclude: $f(x) = -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\sin[(2k-1)\pi x]$. ◁

## Fourier series and even/odd extensions

### Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big[a_n\cos\Big(\frac{n\pi x}{L}\Big) + b_n\sin\Big(\frac{n\pi x}{L}\Big)\Big].$$
Since $f$ is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx,\ L = 2,$$
$$b_n = \int_0^2 (2 - x)\,\sin\Big(\frac{n\pi x}{2}\Big)\,dx.a$$

## Fourier series and even/odd extensions

### Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2\int_0^2 \sin\Big(\frac{n\pi x}{2}\Big)\,dx - \int_0^2 x\,\sin\Big(\frac{n\pi x}{2}\Big)\,dx.$

$$\int \sin\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{-2}{n\pi}\,\cos\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

$$I = \int x \sin\Big(\frac{n\pi x}{2}\Big)\,dx, \qquad \begin{cases} u = x, & v' = \sin\Big(\frac{n\pi x}{2}\Big) \\ u' = 1, & v = -\frac{2}{n\pi}\,\cos\Big(\frac{n\pi x}{2}\Big) \end{cases}$$

$$I = \frac{-2x}{n\pi}\,\cos\Big(\frac{n\pi x}{2}\Big) - \int \Big(\frac{-2}{n\pi}\Big)\,\cos\Big(\frac{n\pi x}{2}\Big)\,dx.$$

## Fourier series and even/odd extensions

### Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \frac{-2x}{n\pi}\,\cos\Big(\frac{n\pi x}{2}\Big) - \int \Big(\frac{-2}{n\pi}\Big)\,\cos\Big(\frac{n\pi x}{2}\Big)\,dx.$

$I = -\frac{2x}{n\pi}\,\cos\Big(\frac{n\pi x}{2}\Big) + \Big(\frac{2}{n\pi}\Big)^2\,\sin\Big(\frac{n\pi x}{2}\Big).$ So, we get

$$b_n = 2\Big[\frac{-2}{n\pi}\,\cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 + \Big[\frac{2x}{n\pi}\,\cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big(\frac{2}{n\pi}\Big)^2\,\sin\Big(\frac{n\pi x}{2}\Big)\Big|_0^2$$

$$b_n = \frac{-4}{n\pi}\big[\cos(n\pi) - 1\big] + \Big[\frac{4}{n\pi}\cos(n\pi) - 0\Big] \quad \Rightarrow \quad b_n = \frac{4}{n\pi}.$$

We conclude: $f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,\sin\Big(\frac{n\pi x}{2}\Big).$ ◁

## Fourier series and even/odd extensions

### Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[ a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since $f$ is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\,dx = \int_0^2 (2-x)\,dx = \frac{\text{base x height}}{2} \quad \Rightarrow \quad a_0 = 2.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L} \int_0^L f(x) \cos\Big(\frac{n\pi x}{L}\Big)\,dx, \;\; L = 2,$$

$$a_n = \int_0^2 (2-x) \cos\Big(\frac{n\pi x}{2}\Big)\,dx.$$

## Fourier series and even/odd extensions

### Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2 \int_0^2 \cos\Big(\frac{n\pi x}{2}\Big)\,dx - \int_0^2 x \cos\Big(\frac{n\pi x}{2}\Big)\,dx.$

$$\int \cos\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

$$I = \int x \cos\Big(\frac{n\pi x}{2}\Big)\,dx, \qquad \begin{cases} u = x, & v' = \cos\Big(\frac{n\pi x}{2}\Big) \\ u' = 1, & v = \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) \end{cases}$$

$$I = \frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) - \int \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\,dx.$$

## Fourier series and even/odd extensions

### Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \int \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) dx.$

$$I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right).$$ So, we get

$$a_n = 2\left[\frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 - \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 - \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right)\Big|_0^2$$

$$a_n = 0 - 0 - \frac{4}{n^2\pi^2}\left[\cos(n\pi) - 1\right] \quad \Rightarrow \quad a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n].$$

## Fourier series and even/odd extensions

### Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0$, $a_0 = 2$, $a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n]$.

If $n = 2k$, then $a_{2k} = \frac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right] = 0.$

If $n = 2k - 1$, then we obtain

$$a_{(2k-1)} = \frac{4}{(2k-1)^2\pi^2}\left[1 - (-1)^{2k-1}\right] = \frac{8}{(2k-1)^2\pi^2}.$$

We conclude: $f(x) = 1 + \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos\left(\frac{(2k-1)\pi x}{2}\right).$ ◁