Differential linear systems (Sect. 5.4, 5.6, 5.7)

- $n \times n$ linear differential systems (5.4).
- Constant coefficients homogenoues systems (5.6).
- Examples: 2×2 linear systems (5.6).
- Phase portraits for 2×2 systems (5.7).

$n \times n$ linear differential systems (5.4).

Definition

An *$n \times n$ linear differential system* is a the following: Given an $n \times n$ matrix-valued function A, and an n-vector-valued function $\mathbf{b}$, find an n-vector-valued function $\mathbf{x}$ solution of

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t).$$

The system above is called *homogeneous* iff holds $\mathbf{b} = 0$.

Recall:

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad \mathbf{b}(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix}, \quad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t) \quad \Leftrightarrow \quad \begin{aligned} x_1' &= a_{11}(t)\,x_1 + \cdots + a_{1n}(t)\,x_n + b_1(t) \\ &\;\vdots \\ x_n' &= a_{n1}(t)\,x_1 + \cdots + a_{nn}(t)\,x_n + b_n(t). \end{aligned}$$

$n \times n$ linear differential systems (5.4).

Example

Find the explicit expression for the linear system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ in the case that

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad \mathbf{b}(t) = \begin{bmatrix} e^t \\ 2e^{3t} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: The 2×2 linear system is given by

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^t \\ 2e^{3t} \end{bmatrix}.$$

That is,

$$x_1'(t) = x_1(t) + 3x_2(t) + e^t,$$
$$x_2'(t) = 3x_1(t) + x_2(t) + 2e^{3t}.$$

$\triangleleft$

$n \times n$ linear differential systems (5.4).

Remark: Derivatives of vector-valued functions are computed component-wise.

$$\mathbf{x}'(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}' = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

Example

Compute $\mathbf{x}'$ for $\mathbf{x}(t) = \begin{bmatrix} e^{2t} \\ \sin(t) \\ \cos(t) \end{bmatrix}$.

Solution:

$$\mathbf{x}'(t) \begin{bmatrix} e^{2t} \\ \sin(t) \\ \cos(t) \end{bmatrix}' = \begin{bmatrix} 2e^{2t} \\ \cos(t) \\ -\sin(t) \end{bmatrix}.$$

$\triangleleft$

Differential linear systems (Sect. 5.4, 5.6, 5.7)

- $n \times n$ linear differential systems (5.4).
- **Constant coefficients homogenoues systems (5.6).**
- Examples: 2×2 linear systems (5.6).
- Phase portraits for 2×2 systems (5.7).

Constant coefficients homogenoues systems (5.6).

Remarks:

- Given an $n \times n$ matrix $A(t)$, n-vector $\mathbf{b}(t)$, find $\mathbf{x}(t)$ solution
$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t).$$
- The system is *homogeneous* iff $\mathbf{b} = 0$, that is,
$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t).$$
- The system has *constant coefficients* iff matrix A does not depend on t, that is,
$$\mathbf{x}'(t) = A\,\mathbf{x}(t) + \mathbf{b}(t).$$
- We study homogeneous, constant coefficient systems, that is,
$$\mathbf{x}'(t) = A\,\mathbf{x}(t).$$

Constant coefficients homogenoues systems (5.6).

Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to the homogeneous, constant coefficients, linear system

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

Remark:

- The differential system for the variable $\mathbf{x}$ is coupled, that is, A is not diagonal.
- We transform the system into a system for a variable $\mathbf{y}$ such that the system for $\mathbf{y}$ is decoupled, that is, $\mathbf{y}'(t) = D\,\mathbf{y}(t)$, where D is a diagonal matrix.
- We solve for $\mathbf{y}(t)$ and we transform back to $\mathbf{x}(t)$.

Differential linear systems (Sect. 5.4, 5.6, 5.7)

- $n \times n$ linear differential systems (5.4).
- Constant coefficients homogenoues systems (5.6).
- **Examples: 2×2 linear systems (5.6).**
- Phase portraits for 2×2 systems (5.7).

Examples: 2×2 linear systems (5.6).

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: Find eigenvalues and eigenvectors of A. We found that:

$$\lambda_1 = 4, \quad \mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_2 = -2, \quad \mathbf{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Fundamental solutions are

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

The general solution is $\mathbf{x}(t) = c_1\, \mathbf{x}^{(1)}(t) + c_2\, \mathbf{x}^{(2)}(t)$, that is,

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}, \qquad c_1, c_2 \in \mathbb{R}. \qquad \triangleleft$$

Examples: 2×2 linear systems (5.6).

Example

Verify that $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$, and $\mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$ are solutions to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: We compute $\mathbf{x}^{(1)\prime}$ and then we compare it with $A\mathbf{x}^{(1)}$,

$$\mathbf{x}^{(1)\prime}(t) = \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix}' = \begin{bmatrix} 4e^{4t} \\ 4e^{4t} \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \quad \Rightarrow \quad \mathbf{x}^{(1)\prime} = 4\mathbf{x}^{(1)}.$$

$$A\mathbf{x}^{(1)} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} e^{4t} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \Rightarrow A\mathbf{x}^{(1)} = 4\mathbf{x}^{(1)}.$$

We conclude that $\mathbf{x}^{(1)\prime} = A\mathbf{x}^{(1)}$.

Examples: 2×2 linear systems (5.6).

Example

Verify that $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$, and $\mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$ are solutions to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: We compute $\mathbf{x}^{(2)\prime}$ and then we compare it with $A\mathbf{x}^{(2)}$,

$$\mathbf{x}^{(2)\prime} = \begin{bmatrix} -e^{-2t} \\ e^{-2t} \end{bmatrix}' = \begin{bmatrix} 2e^{-2t} \\ -2e^{-2t} \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} \Rightarrow \mathbf{x}^{(2)\prime} = -2\mathbf{x}^{(2)}.$$

$$A\mathbf{x}^{(2)} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-2t} = -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t},$$

So, $A\mathbf{x}^{(2)} = -2\mathbf{x}^{(2)}$. Hence, $\mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)}$. $\triangleleft$

Examples: 2×2 linear systems (5.6).

Example

Solve the IVP $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: The general solution: $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$.

The initial condition is,

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We need to solve the linear system

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, hence $\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$. $\triangleleft$

Constant coefficients homogenoues systems (5.6).

Proof: Since A is diagonalizable, we know that $A = PDP^{-1}$, with

$$P = \big[\mathbf{v}_1, \cdots, \mathbf{v}_n\big], \qquad D = \mathrm{diag}\big[\lambda_1, \cdots, \lambda_n\big].$$

Equivalently, $P^{-1}AP = D$. Multiply $\mathbf{x}' = A\,\mathbf{x}$ by P^{-1} on the left

$$P^{-1}\mathbf{x}'(t) = P^{-1}A\,\mathbf{x}(t) \quad \Leftrightarrow \quad \big(P^{-1}\mathbf{x}\big)' = \big(P^{-1}AP\big)\,\big(P^{-1}\mathbf{x}\big).$$

Introduce the new unknown $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, then

$$\mathbf{y}'(t) = D\,\mathbf{y}(t) \;\Leftrightarrow\; \begin{cases} y_1'(t) = \lambda_1\,y_1(t), \\ \quad\vdots \\ y_n'(t) = \lambda_n\,y_n(t), \end{cases} \quad \Rightarrow \; \mathbf{y}(t) = \begin{bmatrix} c_1\,e^{\lambda_1 t} \\ \vdots \\ c_n\,e^{\lambda_n t} \end{bmatrix}.$$

Constant coefficients homogenoues systems (5.6).

Proof: Recall: $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, and $\mathbf{y}(t) = \begin{bmatrix} c_1\,e^{\lambda_1 t} \\ \vdots \\ c_n\,e^{\lambda_n t} \end{bmatrix}$.

Transform back to $\mathbf{x}(t)$, that is,

$$\mathbf{x}(t) = P\,\mathbf{y}(t) = \big[\mathbf{v}_1, \cdots, \mathbf{v}_n\big] \begin{bmatrix} c_1\,e^{\lambda_1 t} \\ \vdots \\ c_n\,e^{\lambda_n t} \end{bmatrix}$$

We conclude: $\mathbf{x}(t) = c_1\mathbf{v}_1\,e^{\lambda_1 t} + \cdots + c_n\mathbf{v}_n\,e^{\lambda_n t}$. □

Remark:

- $A\,\mathbf{v}_i = \lambda_i\mathbf{v}_i$.
- The eigenvalues and eigenvectors of A are crucial to solve the differential linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$.

Differential linear systems (Sect. 5.4, 5.6, 5.7)

- $n \times n$ linear differential systems (5.4).
- Constant coefficients homogenoues systems (5.6).
- Examples: 2×2 linear systems (5.6).
- **Phase portraits for 2×2 systems (5.7).**

Phase portraits for 2×2 systems (5.7).

Remark:

- There are two main types of graphs for solutions of 2×2 linear systems:
  (i) The graphs of the vector components;
  (ii) The phase portrait.
- Case (i): Express the solution in vector components $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, and graph x_1 and x_2 as functions of t.
  (Recall the solution in the IVP of the previous Example: $x_1(t) = 3\,e^{4t} - e^{-2t}$ and $x_2(t) = 3\,e^{4t} + e^{-2t}$.)
- Case (ii): Express the solution as a vector-valued function,
  $$\mathbf{x}(t) = c_1\,\mathbf{v}_1\,e^{\lambda_1 t} + c_2\,\mathbf{v}_2\,e^{\lambda_2 t},$$
  and plot the vector $\mathbf{x}(t)$ for different values of t.
- Case (ii) is called a *phase portrait*.

Phase portraits for 2×2 systems (5.7).

Example

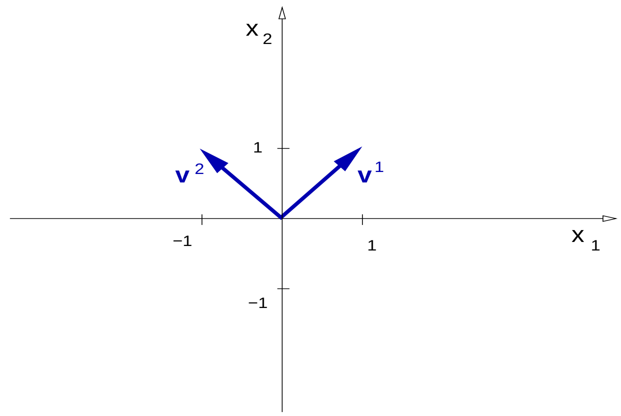
Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:
We start plotting the vectors

$$\mathbf{v}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\mathbf{v}^2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$



Phase portraits for 2×2 systems (5.7).

Example

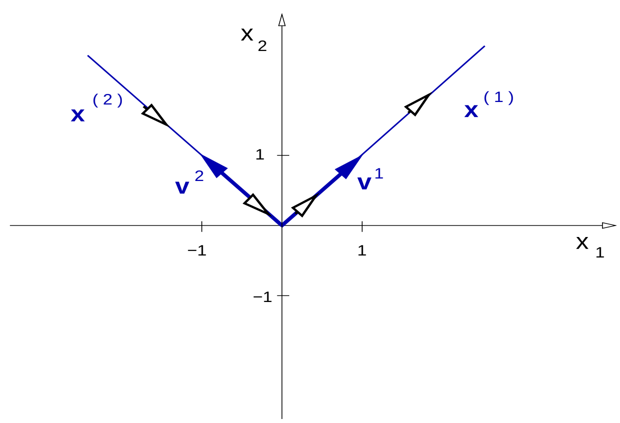
Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:
We now plot the functions

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t},$$

$$\mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$



Phase portraits for 2×2 systems (5.7).

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

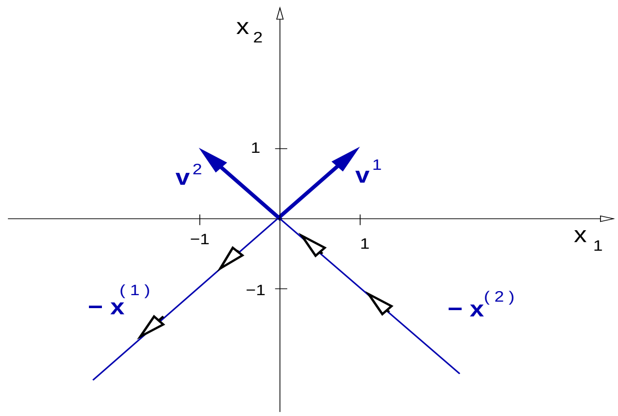
$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:

We now plot the functions

$$-\mathbf{x}^{(1)} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t},$$

$$-\mathbf{x}^{(2)} = -\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$



Phase portraits for 2×2 systems (5.7).

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

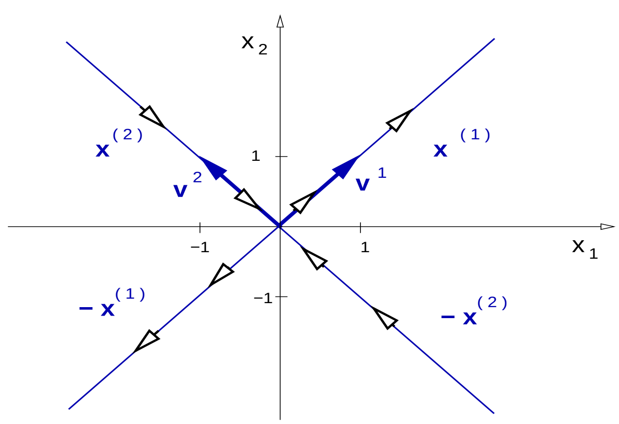
$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:

We now plot the four functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

$$\mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)}.$$



Phase portraits for 2×2 systems (5.7).

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

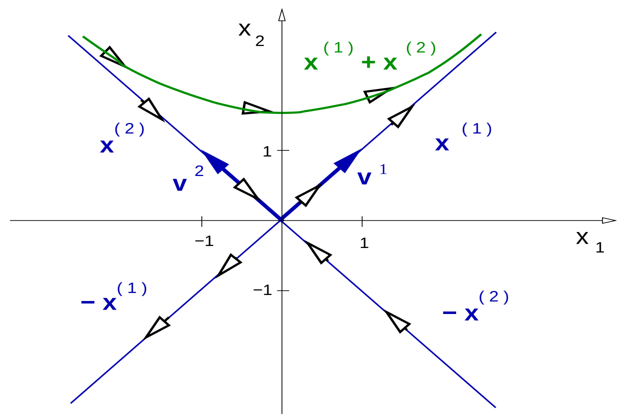
Solution:

We now plot the four functions

$\mathbf{x}^{(1)}, \ -\mathbf{x}^{(1)}, \ \mathbf{x}^{(2)}, \ -\mathbf{x}^{(2)},$

and $\mathbf{x}^{(1)} + \mathbf{x}^{(2)}$,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$



Phase portraits for 2×2 systems (5.7).

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

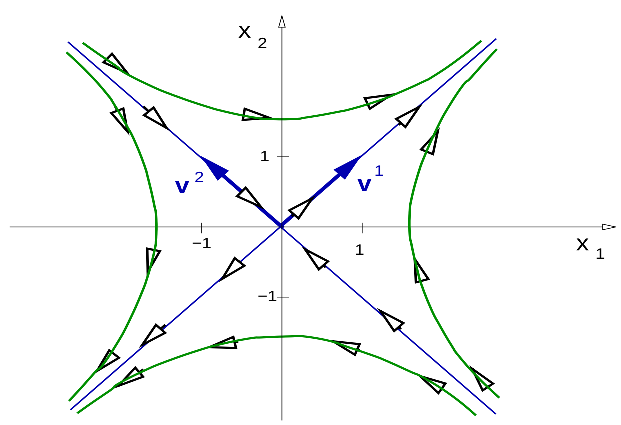
Solution:

We now plot the eight functions

$\mathbf{x}^{(1)}, \ -\mathbf{x}^{(1)}, \ \mathbf{x}^{(2)}, \ -\mathbf{x}^{(2)},$

$\mathbf{x}^{(1)} + \mathbf{x}^{(2)}, \quad -\mathbf{x}^{(1)} + \mathbf{x}^{(2)},$

$\mathbf{x}^{(1)} - \mathbf{x}^{(2)}, \quad -\mathbf{x}^{(1)} - \mathbf{x}^{(2)}.$



Phase portraits for 2×2 systems (5.7).

Problem:
Case (a): Consider a 2×2 matrix A having two different, real eigenvalues $\lambda_1 \neq \lambda_2$, so A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions).

Given a solution $\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}$, to $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, plot different solution vectors $\mathbf{x}(t)$ on the plane as function of t for different choices of the constants c_1 and c_2.

The plots are different depending on the eigenvalues signs.
We have the following three sub-cases:

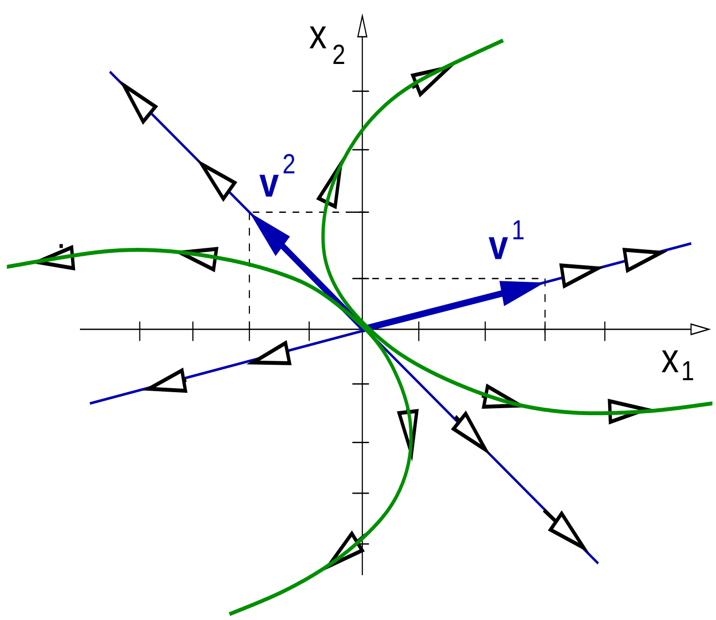
(i) $0 < \lambda_2 < \lambda_1$, both positive;

(ii) $\lambda_2 < 0 < \lambda_1$, one positive the other negative;

(iii) $\lambda_2 < \lambda_1 < 0$, both negative.

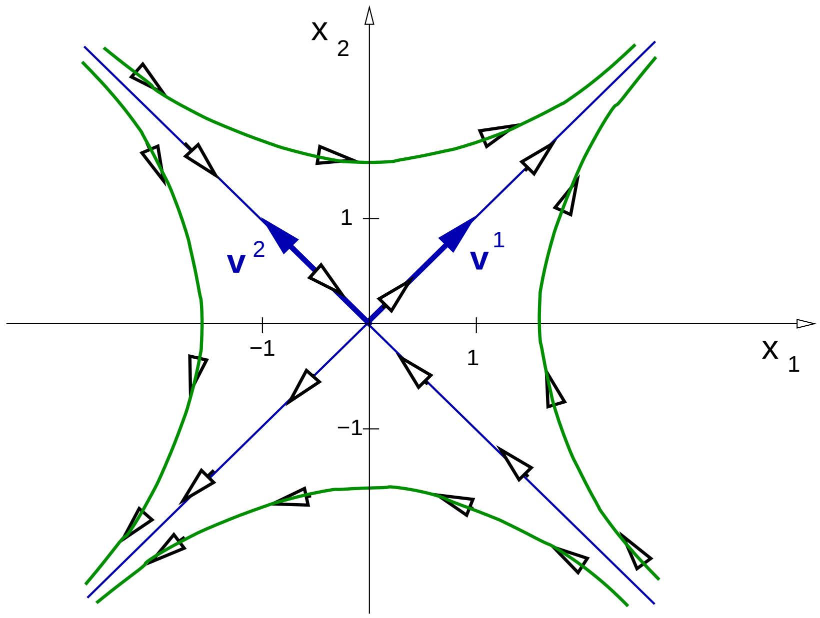
Phase portraits for 2×2 systems (5.7).

Phase portrait: Case (a), two different, real eigenvalues $\lambda_1 \neq \lambda_2$, sub-case $0 < \lambda_2 < \lambda_1$, both eigenvalue positive.



Phase portraits for 2×2 systems.

Phase portrait: Case (a), two different, real eigenvalues $\lambda_1 \neq \lambda_2$, sub-case $\lambda_2 < 0 < \lambda_1$, one eigenvalue positive the other negative.



Phase portraits for 2×2 systems (5.7).

Phase portrait: Case (a), two different, real eigenvalues $\lambda_1 \neq \lambda_2$, sub-case $\lambda_2 < \lambda_1 < 0$, both eigenvalues negative.