Review Exam 3.

- Sections 4.1-4.5 in Lecture Notes.
- 60 minutes.
- 7 problems.
- 70 grade attempts. (10 attempts per problem.)
- No partial grading. (Exceptions allowed, ask you TA.)
- Integration table included.
- Complete Laplace Transform table included.

Computing inverse Laplace Transforms.

Example

Find the inverse Laplace Transform of $F(s) = \dfrac{e^{-2s}s}{(s-3)^2 + 25}$.

Solution: We start rewriting function F,

$$F(s) = e^{-2s}\,\frac{s}{(s-3)^2 + 5^2} = e^{-2s}\,\frac{[(s-3) + 3]}{(s-3)^2 + 5^2}$$

$$F(s) = e^{-2s}\frac{(s-3)}{(s-3)^2 + 5^2} + e^{-2s}\,\frac{1}{5}\,\frac{5\,(3)}{(s-3)^2 + 5^2}$$

$$F(s) = e^{-2s}\mathcal{L}[\cos(5t)](s-3) + \frac{3}{5}\,e^{-2s}\,\mathcal{L}[\sin(5t)](s-3)$$

Recall (14): $\mathcal{L}[f(t)](s-c) = \mathcal{L}[e^{ct}\,f(t)]$.

$$F(s) = e^{-2s}\mathcal{L}[e^{3t}\cos(5t)] + \frac{3}{5}\,e^{-2s}\,\mathcal{L}[e^{3t}\,\sin(5t)].$$

Computing inverse Laplace Transforms.

Example

Find the inverse Laplace Transform of $F(s) = \dfrac{e^{-2s}\,s}{(s-3)^2+25}$.

Solution:
Recall: $F(s) = e^{-2s}\mathcal{L}[e^{3t}\cos(5t)] + \dfrac{3}{5}\,e^{-2s}\,\mathcal{L}[e^{3t}\,\sin(5t)]$.

Recall (13): $e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u(t-c)\,f(t-c)]$.

$$F(s) = \mathcal{L}[u(t-2)\,e^{3(t-2)}\cos(5(t-2))] + \frac{3}{5}\,\mathcal{L}[u(t-2)\,e^{3(t-2)}\sin(5(t-2))].$$

$$f(t) = u(t-2)\,e^{3(t-2)}\Big[\cos(5(t-2)) + \frac{3}{5}\,\sin(5(t-2))\Big]. \quad \triangleleft$$

Computing Laplace Transforms of discontinuous functions.

Example

Find the LT of $f(t) = \begin{cases} \dfrac{t}{2} & 0 \leqslant t \leqslant 6, \\ 3 & t \geqslant 6. \end{cases}$.

Solution: We need to rewrite the function f in terms of functions that apear in the LT table. We need a box function for the first part, and a step function for the second part.

$$f(t) = \frac{t}{2}\big[u(t) - u(t-6)\big] + 3\,u(t-6).$$

$$f(t) = \frac{t}{2}\,u(t) + \Big(-\frac{t}{2} + 3\Big)\,u(t-6) = \frac{t}{2}\,u(t) + \frac{1}{2}(-t+6)\,u(t-6).$$

$$f(t) = \frac{1}{2}\big[t\,u(t) - (t-6)\,u(t-6)\big].$$

Computing Laplace Transforms of discontinuous functions.

Example

Find the LT of $f(t) = \begin{cases} \dfrac{t}{2} & 0 \leqslant t \leqslant 6, \\ 3 & t \geqslant 6. \end{cases}$.

Solution: Recall: $f(t) = \dfrac{1}{2}\big[t\, u(t) - (t-6)\, u(t-6)\big]$.

$$\mathcal{L}[f(t)] = \frac{1}{2}\big(\mathcal{L}[t\, u(t)] - \mathcal{L}[(t-6)\, u(t-6)]\big),$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\big(\mathcal{L}[t] - e^{-6s}\mathcal{L}[t]\big),$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\big(\frac{1}{s^2} - e^{-6s}\frac{1}{s^2}\big),$$

We conclude that $\mathcal{L}[f(t)] = \dfrac{1}{2s^2}\big(1 - e^{-6s}\big)$. $\triangleleft$

Solving IVP with generalized sources.

Example

Use Laplace Transform to find y solution of

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: Compute the LT of the equation,

$$\mathcal{L}[y''] - 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[\delta(t-2)] = e^{-2s}$$

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0), \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0).$$

$$(s^2 - 2s + 2)\,\mathcal{L}[y] - s\,y(0) - y'(0) + 2\,y(0) = e^{-2s}$$

$$(s^2 - 2s + 2)\,\mathcal{L}[y] - s - 1 = e^{-2s}$$

$$\mathcal{L}[y] = \frac{(s+1)}{(s^2 - 2s + 2)} + \frac{1}{(s^2 - 2s + 2)}\, e^{-2s}.$$

Solving IVP with generalized sources.

Example

Use Laplace Transform to find y solution of

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2 - 2s + 2)} + \dfrac{1}{(s^2 - 2s + 2)}\, e^{-2s}$.

$$s^2 - 2s + 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4-8}\big], \quad \text{complex roots.}$$

$$s^2 - 2s + 2 = (s^2 - 2s + 1) - 1 + 2 = (s-1)^2 + 1.$$

$$\mathcal{L}[y] = \frac{s+1}{(s-1)^2 + 1} + \frac{1}{(s-1)^2 + 1}\, e^{-2s}$$

$$\mathcal{L}[y] = \frac{(s-1+1)+1}{(s-1)^2 + 1} + \frac{1}{(s-1)^2 + 1}\, e^{-2s}$$

Solving IVP with generalized sources.

Example

Use Laplace Transform to find y solution of

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s-1)+2}{(s-1)^2 + 1} + \dfrac{1}{(s-1)^2 + 1}\, e^{-2s}$,

$$\mathcal{L}[y] = \frac{(s-1)}{(s-1)^2 + 1} + 2\,\frac{1}{(s-1)^2 + 1} + e^{-2s}\,\frac{1}{(s-1)^2 + 1},$$

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2}, \qquad \mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2},$$

$$\mathcal{L}[y] = \mathcal{L}[\cos(t)]\big|_{(s-1)} + 2\,\mathcal{L}[\sin(t)]\big|_{(s-1)} + e^{-2s}\,\mathcal{L}[\sin(t)]\big|_{(s-1)}.$$

Solving IVP with generalized sources.

Example

Use Laplace Transform to find y solution of

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: Recall:

$$\mathcal{L}[y] = \mathcal{L}[\cos(t)]\big|_{(s-1)} + 2\,\mathcal{L}[\sin(t)]\big|_{(s-1)} + e^{-2s}\,\mathcal{L}[\sin(t)]\big|_{(s-1)}$$

and $\mathcal{L}[f(t)]\big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]$. Therefore,

$$\mathcal{L}[y] = \mathcal{L}[e^t\,\cos(t)] + 2\,\mathcal{L}[e^t\,\sin(t)] + e^{-2s}\,\mathcal{L}[e^t\,\sin(t)].$$

Also recall: $e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]$. Therefore,

$$\mathcal{L}[y] = \mathcal{L}[e^t\,\cos(t)] + 2\,\mathcal{L}[e^t\,\sin(t)] + \mathcal{L}[u_2(t)\,e^{(t-2)}\,\sin(t-2)].$$

$$y(t) = \big[\cos(t) + 2\sin(t)\big]\,e^t + u_2(t)\,\sin(t-2)\,e^{(t-2)}. \qquad \triangleleft$$

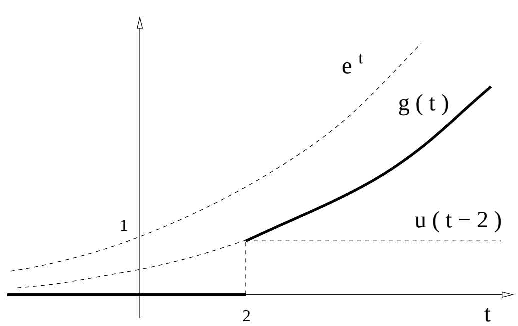
Solving IVP with discontinuous sources.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution:



Express g using step functions,

$$g(t) = u_2(t)\,e^{(t-2)}.$$

$$\mathcal{L}[u_c(t)\,f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)].$$

Therefore,

$$\mathcal{L}[g(t)] = e^{-2s}\mathcal{L}[e^t].$$

We obtain: $\mathcal{L}[g(t)] = \dfrac{e^{-2s}}{(s-1)}$.

Solving IVP with discontinuous sources.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution: Recall: $\mathcal{L}[g(t)] = \dfrac{e^{-2s}}{(s-1)}$.

$$\mathcal{L}[y''] + 3\,\mathcal{L}[y] = \mathcal{L}[g(t)] = \frac{e^{-2s}}{(s-1)}.$$

$$(s^2+3)\,\mathcal{L}[y] = \frac{e^{-2s}}{(s-1)} \quad \Rightarrow \quad \mathcal{L}[y] = e^{-2s}\,\frac{1}{(s-1)(s^2+3)}.$$

$$H(s) = \frac{1}{(s-1)(s^2+3)} = \frac{a}{(s-1)} + \frac{(bs+c)}{(s^2+3)}$$

$$1 = a(s^2+3) + (bs+c)(s-1)$$

Solving IVP with discontinuous sources.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution: Recall: $1 = a(s^2+3) + (bs+c)(s-1)$.

$$1 = as^2 + 3a + bs^2 + cs - bs - c$$

$$1 = (a+b)\,s^2 + (c-b)\,s + (3a-c)$$

$$a+b=0, \quad c-b=0, \quad 3a-c=1.$$

$$a=-b, \quad c=b, \quad -3b-b=1 \quad \Rightarrow \quad b=-\frac{1}{4},\ a=\frac{1}{4},\ c=-\frac{1}{4}.$$

$$H(s) = \frac{1}{4}\Big[\frac{1}{s-1} - \frac{s+1}{s^2+3}\Big].$$

Solving IVP with discontinuous sources.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution: Recall: $H(s) = \frac{1}{4}\Big[\frac{1}{s-1} - \frac{s+1}{s^2+3}\Big]$, $\mathcal{L}[y] = e^{-2s}\,H(s)$.

$$H(s) = \frac{1}{4}\Big[\frac{1}{s-1} - \frac{s}{s^2+3} - \frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{s^2+3}\Big],$$

$$H(s) = \frac{1}{4}\Big[\mathcal{L}[e^t] - \mathcal{L}\big[\cos\big(\sqrt{3}\,t\big)\big] - \frac{1}{\sqrt{3}}\,\mathcal{L}\big[\sin\big(\sqrt{3}\,t\big)\big]\Big].$$

$$H(s) = \mathcal{L}\Big[\frac{1}{4}\Big(e^t - \cos\big(\sqrt{3}\,t\big) - \frac{1}{\sqrt{3}}\,\sin\big(\sqrt{3}\,t\big)\Big)\Big].$$

Solving IVP with discontinuous sources.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution: Recall: $H(s) = \mathcal{L}\Big[\frac{1}{4}\Big(e^t - \cos\big(\sqrt{3}\,t\big) - \frac{1}{\sqrt{3}}\,\sin\big(\sqrt{3}\,t\big)\Big)\Big]$.

$$h(t) = \frac{1}{4}\Big(e^t - \cos\big(\sqrt{3}\,t\big) - \frac{1}{\sqrt{3}}\,\sin\big(\sqrt{3}\,t\big)\Big), \quad H(s) = \mathcal{L}[h(t)].$$

$$\mathcal{L}[y] = e^{-2s}\,H(s) = e^{-2s}\,\mathcal{L}[h(t)] = \mathcal{L}[u_2(t)\,h(t-2)].$$

We conclude: $y(t) = u_2(t)\,h(t-2)$. Equivalently,

$$y(t) = \frac{u_2(t)}{4}\,\Big[e^{(t-2)} - \cos\big(\sqrt{3}\,(t-2)\big) - \frac{1}{\sqrt{3}}\,\sin\big(\sqrt{3}\,(t-2)\big)\Big]. \quad \lhd$$

Laplace Transform and convolutions.

Example

Use convolutions to find f satisfying $\mathcal{L}[f(t)] = \dfrac{e^{-2s}}{(s-1)(s^2+3)}$.

Solution: One way to solve this is with the splitting

$$\mathcal{L}[f(t)] = e^{-2s}\,\frac{1}{(s^2+3)}\,\frac{1}{(s-1)} = e^{-2s}\,\frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{(s^2+3)}\,\frac{1}{(s-1)},$$

$$\mathcal{L}[f(t)] = e^{-2s}\,\frac{1}{\sqrt{3}}\,\mathcal{L}[\sin\big(\sqrt{3}\,t\big)]\,\mathcal{L}[e^t]$$

$$\mathcal{L}[f(t)] = \frac{1}{\sqrt{3}}\mathcal{L}[u_2(t)\,\sin\big(\sqrt{3}\,(t-2)\big)]\,\mathcal{L}[e^t].$$

$$f(t) = \frac{1}{\sqrt{3}}\int_0^t u_2(\tau)\,\sin\big(\sqrt{3}\,(\tau-2)\big)\,e^{(t-\tau)}\,d\tau. \qquad \lhd$$

Definition of the Laplace Transform.

Example

Use the definition of the LT to find the LT of $f(t) = \cosh(t)$.

Solution: Recall that $\cosh(t) = (e^t + e^{-t})/2$, and that

$$\mathcal{L}[\cosh(t)] = \int_0^\infty e^{-st}\,\frac{(e^t + e^{-t})}{2}\,dt$$

$$\mathcal{L}[\cosh(t)] = \lim_{N\to\infty}\frac{1}{2}\int_0^N \big(e^{-(s-1)t} + e^{-(s+1)t}\big)\,dt.$$

$$\mathcal{L}[\cosh(t)] = \lim_{N\to\infty}\frac{1}{2}\Big[-\frac{e^{-(s-1)t}}{(s-1)} - \frac{e^{-(s+1)t}}{(s+1)}\Big]\Big|_0^N.$$

$$\mathcal{L}[\cosh(t)] = \frac{1}{2}\Big[\frac{1}{(s-1)} + \frac{1}{(s+1)}\Big] = \frac{1}{2}\,\frac{(s+1)+(s-1)}{(s^2-1)}.$$

We conclude: $\mathcal{L}[\cosh(t)] = \dfrac{s}{s^2-1}$. $\lhd$

Solving IVP with impulsive forces.

Example

(Sect 6.5, ∼ Probl.7) Find the solution to the initial value problem

$$y'' + y = \delta(t-\pi)\cos(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Compute the Laplace Transform of the equation,

$$\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)\cos(t)]$$

To compute the right-hand side above, we need the definition of the LT. Given any smooth function f and a constant c, holds

$$\mathcal{L}[\delta(t-c)f(t)] = \int_0^{\infty} e^{-st}\, f(t)\,\delta(t-c)\,dt = \big[e^{-st}f(t)\big]\Big|_{t=c}$$

We have used that $\displaystyle\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,g(t)\,dt = g(c).$

We obtain the formula: $\mathcal{L}[\delta(t-c)f(t)] = f(c)\,e^{-cs}.$

Solving IVP with impulsive forces.

Example

(Sect 6.5, ∼ Probl.7) Find the solution to the initial value problem

$$y'' + y = \delta(t-\pi)\cos(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Recall: $\mathcal{L}[\delta(t-c)f(t)] = f(c)\,e^{-cs}$. Hence

$$s^2\,\mathcal{L}[y] + \mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)\cos(t)] = \cos(\pi)e^{-\pi s} = -e^{-\pi s}$$

$$\mathcal{L}[y] = -e^{-\pi s}\,\frac{1}{s^2+1} = -e^{-\pi s}\,\mathcal{L}\big[\sin(t)\big].$$

Recall the property (13): $e^{-cs}\mathcal{L}[f(t)] = \mathcal{L}\big[u(t-c)\,f(t-c)\big]$;

$$\mathcal{L}[y] = -\mathcal{L}\big[u(t-\pi)\,\sin(t-\pi)\big] \;\Rightarrow\; y(t) = -u(t-\pi)\,\sin(t-\pi).$$

◁

Laplace Transform and convolutions.

Example

Given any function $g(t)$ with Laplace transform $G(s) = \mathcal{L}[g(t)]$, find the function f satisfying $\mathcal{L}[f(t)] = \dfrac{e^{-2s}}{(s^2+3)}\, G(s)$.

Solution: One way to solve this is with the splitting

$$\mathcal{L}[f(t)] = e^{-2s}\,\frac{1}{(s^2+3)}\,G(s) = e^{-2s}\,\frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{(s^2+3)}\,G(s),$$

$$\mathcal{L}[f(t)] = e^{-2s}\,\frac{1}{\sqrt{3}}\,\mathcal{L}\big[\sin\big(\sqrt{3}\,t\big)\big]\,\mathcal{L}[g(t)]$$

$$\mathcal{L}[f(t)] = \frac{1}{\sqrt{3}}\mathcal{L}\big[u_2(t)\,\sin\big(\sqrt{3}\,(t-2)\big)\big]\,\mathcal{L}[g(t)].$$

$$f(t) = \frac{1}{\sqrt{3}}\int_0^t u_2(\tau)\,\sin\big(\sqrt{3}\,(\tau-2)\big)\,g(t-\tau)1\,d\tau. \qquad \triangleleft$$

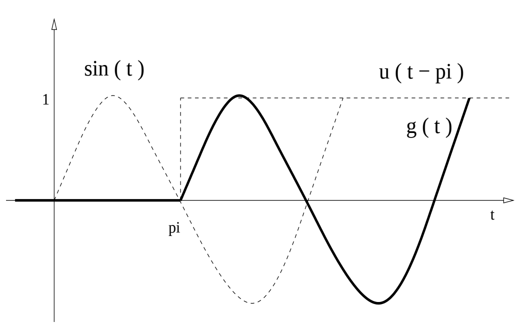
Solving IVP with step functions in the source.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t-\pi), & t \geqslant \pi. \end{cases}$$

Solution:



Express g using step functions,

$$g(t) = u_\pi(t)\,\sin(t-\pi).$$

$$\mathcal{L}[u_c(t)\,f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)].$$

Therefore,

$$\mathcal{L}[g(t)] = e^{-\pi s}\mathcal{L}[\sin(t)].$$

We obtain: $\mathcal{L}[g(t)] = \dfrac{e^{-\pi s}}{s^2+1}$.

Solving IVP with step functions in the source.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t-\pi), & t \geqslant \pi. \end{cases}$$

Solution: $\mathcal{L}[g(t)] = \dfrac{e^{-\pi s}}{s^2+1}$.

$$\mathcal{L}[y''] - 6\,\mathcal{L}[y] = \mathcal{L}[g(t)] = \frac{e^{-\pi s}}{s^2+1}.$$

$$(s^2-6)\,\mathcal{L}[y] = \frac{e^{-\pi s}}{s^2+1} \quad \Rightarrow \quad \mathcal{L}[y] = e^{-\pi s}\,\frac{1}{(s^2+1)(s^2-6)}.$$

$$H(s) = \frac{1}{(s^2+1)(s^2-6)} = \frac{1}{(s^2+1)(s+\sqrt{6})(s-\sqrt{6})}$$

$$H(s) = \frac{a}{(s+\sqrt{6})} + \frac{b}{(s-\sqrt{6})} + \frac{(cs+d)}{(s^2+1)}.$$

Solving IVP with step functions in the source.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t-\pi), & t \geqslant \pi. \end{cases}$$

Solution: $H(s) = \dfrac{a}{(s+\sqrt{6})} + \dfrac{b}{(s-\sqrt{6})} + \dfrac{(cs+d)}{(s^2+1)}$.

$$\frac{1}{(s^2+1)(s+\sqrt{6})(s-\sqrt{6})} = \frac{a}{(s+\sqrt{6})} + \frac{b}{(s-\sqrt{6})} + \frac{(cs+d)}{(s^2+1)}$$

$$1 = a(s-\sqrt{6})(s^2+1) + b(s+\sqrt{6})(s^2+1) + (cs+d)(s^2-6).$$

The solution is: $a = -\dfrac{1}{14\sqrt{6}}$, $b = \dfrac{1}{14\sqrt{6}}$, $c = 0$, $d = -\dfrac{1}{7}$.

Solving IVP with step functions in the source.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t-\pi), & t \geqslant \pi. \end{cases}$$

Solution: $H(s) = \frac{1}{14\sqrt{6}}\Big[-\frac{1}{(s+\sqrt{6})} + \frac{1}{(s-\sqrt{6})} - \frac{2\sqrt{6}}{(s^2+1)}\Big]$.

$$H(s) = \frac{1}{14\sqrt{6}}\Big[-\mathcal{L}\big[e^{-\sqrt{6}\,t}\big] + \mathcal{L}\big[e^{\sqrt{6}\,t}\big] - 2\sqrt{6}\,\mathcal{L}[\sin(t)]\Big]$$

$$H(s) = \mathcal{L}\Big[\frac{1}{14\sqrt{6}}\Big(-e^{-\sqrt{6}\,t} + e^{\sqrt{6}\,t} - 2\sqrt{6}\,\sin(t)\Big)\Big].$$

$$h(t) = \frac{1}{14\sqrt{6}}\Big[-e^{-\sqrt{6}\,t} + e^{\sqrt{6}\,t} - 2\sqrt{6}\,\sin(t)\Big] \;\Rightarrow\; H(s) = \mathcal{L}[h(t)].$$

Solving IVP with step functions in the source.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t-\pi), & t \geqslant \pi. \end{cases}$$

Solution: Recall: $\mathcal{L}[y] = e^{-\pi s}\,H(s)$, where $H(s) = \mathcal{L}[h(t)]$, and

$$h(t) = \frac{1}{14\sqrt{6}}\Big[-e^{-\sqrt{6}\,t} + e^{\sqrt{6}\,t} - 2\sqrt{6}\,\sin(t)\Big].$$

$$\mathcal{L}[y] = e^{-\pi s}\,\mathcal{L}[h(t)] = \mathcal{L}[u_\pi(t)\,h(t-\pi)] \;\Rightarrow\; y(t) = u_\pi(t)\,h(t-\pi).$$

Equivalently:

$$y(t) = \frac{u_\pi(t)}{14\sqrt{6}}\Big[-e^{-\sqrt{6}\,(t-\pi)} + e^{\sqrt{6}\,(t-\pi)} - 2\sqrt{6}\,\sin(t-\pi)\Big]. \quad \triangleleft$$