Generalized sources (Sect. 4.4).

- The Dirac delta generalized function.
- Properties of Dirac's delta.
- Relation between deltas and steps.
- Dirac's delta in Physics.
- The Laplace Transform of Dirac's delta.
- Differential equations with Dirac's delta sources.

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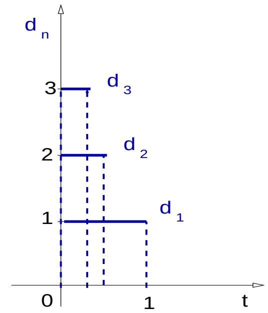
The Dirac delta generalized function.

Definition

Consider the sequence of functions for $n \geqslant 1$,

$$\delta_n(t) = \begin{cases} 0, & t < 0 \\ n, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 0, & t > \dfrac{1}{n}. \end{cases}$$



The Dirac delta generalized function is given by

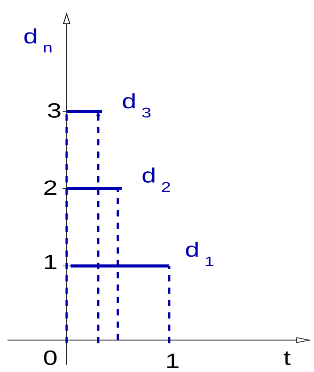
$$\lim_{n\to\infty} \delta_n(t) = \delta(t), \qquad t \in \mathbb{R}.$$

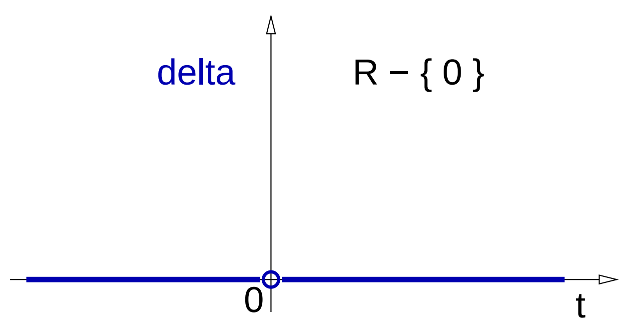
Remarks:

(a) There exist infinitely many sequences δ_n that define the same generalized function δ.

(b) For example, compare with the sequences δ_n in the literature.

The Dirac delta generalized function.





Remarks:

(a) The Dirac δ is a function on the domain $\mathbb{R} - \{0\}$, and $\delta(t) = 0$ for $t \in \mathbb{R} - \{0\}$.

(b) δ at $t = 0$ is not defined, since $\delta(0) = \lim_{n\to\infty} n = +\infty$.

(c) δ is not a function on $\mathbb{R}$.

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Properties of Dirac's delta.

Remark: The Dirac δ is not a function on $\mathbb{R}$.

We define operations on Dirac's δ as limits $n \to \infty$ of the operation on the sequence elements δ_n.

Definition

$$\delta(t-c) = \lim_{n\to\infty} \delta_n(t-c),$$

$$a\,\delta(t) + b\,\delta(t) = \lim_{n\to\infty} \big[a\,\delta_n(t) + b\,\delta_n(t)\big],$$

$$f(t)\,\delta(t) = \lim_{n\to\infty} \big[f(t)\,\delta_n(t)\big],$$

$$\int_a^b \delta(t)\,dt = \lim_{n\to\infty} \int_a^b \delta_n(t)\,dt,$$

$$\mathcal{L}[\delta] = \lim_{n\to\infty} \mathcal{L}[\delta_n].$$

Properties of Dirac's delta.

Theorem

$$\int_{-a}^{a} \delta(t)\,dt = 1, \qquad a > 0.$$

Proof:

$$\int_{-a}^{a} \delta(t)\,dt = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(t)\,dt = \lim_{n\to\infty} \int_{0}^{1/n} n\,dt$$

$$\int_{-a}^{a} \delta(t)\,dt = \lim_{n\to\infty} \Big[n \Big(t\Big|_0^{1/n} \Big) \Big] = \lim_{n\to\infty} \Big[n\,\frac{1}{n} \Big].$$

We conclude: $\displaystyle\int_{-a}^{a} \delta(t)\,dt = 1.$ $\square$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\,dt = f(t_0).$$

Proof: Introduce the change of variable $\tau = t - t_0$,

$$I = \int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\,dt = \int_{-a}^{a} \delta(\tau)\, f(\tau + t_0)\,d\tau,$$

$$I = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(\tau)\, f(\tau + t_0)\,d\tau = \lim_{n\to\infty} \int_{0}^{1/n} n\, f(\tau + t_0)\,d\tau$$

Therefore, $I = \displaystyle\lim_{n\to\infty} n \int_0^{1/n} F'(\tau + t_0)\,d\tau$, where we introduced the primitive $F(t) = \displaystyle\int f(t)\,dt$, that is, $f(t) = F'(t)$.

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: So, $I = \lim_{n\to\infty} n \int_0^{1/n} F'(\tau + t_0)\, d\tau$, with $f(t) = F'(t)$.

$$I = \lim_{n\to\infty} n \left[F(\tau + t_0)\Big|_0^{1/n} \right] = \lim_{n\to\infty} n \left[F\Big(t_0 + \frac{1}{n}\Big) - F(t_0) \right].$$

$$I = \lim_{n\to\infty} \frac{\left[F\big(t_0 + \frac{1}{n}\big) - F(t_0) \right]}{\frac{1}{n}} = F'(t_0) = f(t_0).$$

We conclude: $\displaystyle\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$ $\square$

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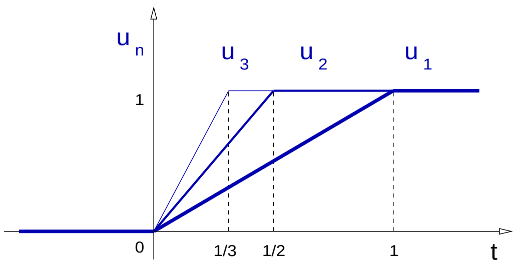
Relation between deltas and steps.

Theorem

The sequence of functions for $n \geqslant 1$,

$$u_n(t) = \begin{cases} 0, & t < 0 \\ nt, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 1, & t > \dfrac{1}{n}. \end{cases}$$



satisfies, for $t \in (-\infty, 0) \cup (0, 1/n) \cup (1/n, \infty)$, both equations,

$$u_n'(t) = \delta_n(t), \qquad \lim_{n\to\infty} u_n(t) = u(t), \qquad t \in \mathbb{R}.$$

Remark:

- If we generalize the notion of derivative as $u'(t) = \lim_{n\to\infty} u_n'(t)$, then holds $u'(t) = \delta(t)$.
- Dirac's delta is a generalized derivative of the step function.

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Dirac's delta in Physics.

Remarks:

(a) Dirac's delta generalized function is useful to describe *impulsive forces* in mechanical systems.

(b) An impulsive force transmits a finite momentum in an infinitely short time.

(c) For example: The momentum transmitted to a pendulum when hit by a hammer. Newton's law of motion says,

$$m\, v'(t) = F(t), \quad \text{with} \quad F(t) = F_0\, \delta(t - t_0).$$

The momentum transfer is:

$$\Delta I = \lim_{\Delta t \to 0} mv(t)\Big|_{t_0 - \Delta t}^{t_0 + \Delta t} = \lim_{\Delta t \to 0} \int_{t_0 - \Delta t}^{t_0 + \Delta t} F(t)\, dt = F_0.$$

That is, $\Delta I = F_0$.

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The Laplace Transform of Dirac's delta.

Recall: The Laplace Transform can be generalized from functions to δ, as follows, $\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)]$.

Theorem

$$\mathcal{L}[\delta(t-c)] = e^{-cs}.$$

Proof:

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)], \qquad \delta_n(t) = n\Big[u(t) - u\big(t-\frac{1}{n}\big)\Big].$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} n\Big(\mathcal{L}[u(t-c)] - \mathcal{L}\Big[u\big(t-c-\frac{1}{n}\big)\Big]\Big)$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} n\Big(\frac{e^{-cs}}{s} - \frac{e^{-(c+\frac{1}{n})s}}{s}\Big) = e^{-cs} \lim_{n\to\infty} \frac{(1-e^{-\frac{s}{n}})}{\big(\frac{s}{n}\big)}.$$

This is a singular limit, $\frac{0}{0}$. Use l'Hôpital rule.

The Laplace Transform of Dirac's delta.

Proof: Recall: $\mathcal{L}[\delta(t-c)] = e^{-cs} \lim_{n\to\infty} \frac{(1-e^{-\frac{s}{n}})}{\big(\frac{s}{n}\big)}$.

$$\lim_{n\to\infty} \frac{(1-e^{-\frac{s}{n}})}{\big(\frac{s}{n}\big)} = \lim_{n\to\infty} \frac{\big(-\frac{s}{n^2}\, e^{-\frac{s}{n}}\big)}{\big(-\frac{s}{n^2}\big)} = \lim_{n\to\infty} e^{-\frac{s}{n}} = 1.$$

We therefore conclude that $\mathcal{L}[\delta(t-c)] = e^{-cs}$. $\square$

Remarks:

(a) This result is consistent with a previous result:
$$\int_{t_0-a}^{t_0+a} \delta(t-t_0)\, f(t)\, dt = f(t_0).$$

(b) $\mathcal{L}[\delta(t-c)] = \int_0^\infty \delta(t-c)\, e^{-st}\, dt = e^{-cs}$.

(c) $\mathcal{L}[\delta(t-c)\, f(t)] = \int_0^\infty \delta(t-c)\, e^{-st}\, f(t)\, dt = e^{-cs}\, f(c)$.

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Differential equations with Dirac's delta sources.

Example

Find the solution y to the initial value problem

$$y'' - y = -20\,\delta(t-3), \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] - \mathcal{L}[y] = -20\,\mathcal{L}[\delta(t-3)]$.

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad (s^2-1)\,\mathcal{L}[y] - s = -20\,e^{-3s},$$

We arrive to the equation $\mathcal{L}[y] = \dfrac{s}{(s^2-1)} - 20\,e^{-3s}\,\dfrac{1}{(s^2-1)}$,

$$\mathcal{L}[y] = \mathcal{L}[\cosh(t)] - 20\,\mathcal{L}[u(t-3)\,\sinh(t-3)],$$

We conclude: $y(t) = \cosh(t) - 20\,u(t-3)\,\sinh(t-3)$. $\triangleleft$

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] + 4\,\mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)] - \mathcal{L}[\delta(t-2\pi)]$,

$$(s^2+4)\,\mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{e^{-\pi s}}{(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+4)},$$

that is, $\mathcal{L}[y] = \dfrac{e^{-\pi s}}{2}\,\dfrac{2}{(s^2+4)} - \dfrac{e^{-2\pi s}}{2}\,\dfrac{2}{(s^2+4)}$.

Recall: $e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u(t-c)\,f(t-c)]$. Therefore,

$$\mathcal{L}[y] = \frac{1}{2}\,\mathcal{L}\Big[u(t-\pi)\,\sin\big[2(t-\pi)\big]\Big] - \frac{1}{2}\,\mathcal{L}\Big[u(t-2\pi)\,\sin\big[2(t-2\pi)\big]\Big].$$

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Recall:

$$\mathcal{L}[y] = \frac{1}{2}\,\mathcal{L}\Big[u(t-\pi)\,\sin\big[2(t-\pi)\big]\Big] - \frac{1}{2}\,\mathcal{L}\Big[u(t-2\pi)\,\sin\big[2(t-2\pi)\big]\Big].$$

This implies that,

$$y(t) = \frac{1}{2}\,u(t-\pi)\,\sin\big[2(t-\pi)\big] - \frac{1}{2}\,u(t-2\pi)\,\sin\big[2(t-2\pi)\big],$$

We conclude: $y(t) = \dfrac{1}{2}\,\big[u(t-\pi) - u(t-2\pi)\big]\,\sin(2t)$. $\lhd$