The Laplace Transform of step functions (Sect. 4.3).

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.

Overview and notation.

Overview: The Laplace Transform method can be used to solve constant coefficients differential equations with *discontinuous source functions*.

Notation:
If $\mathcal{L}[f(t)] = F(s)$, then we denote $\mathcal{L}^{-1}[F(s)] = f(t)$.

Remark: One can show that for a particular type of functions f, that includes all functions we work with in this Section, the notation above is well-defined.

Example
From the Laplace Transform table we know that $\mathcal{L}\big[e^{at}\big] = \dfrac{1}{s-a}$.

Then also holds that $\mathcal{L}^{-1}\Big[\dfrac{1}{s-a}\Big] = e^{at}$. $\triangleleft$

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The definition of a step function.

Definition

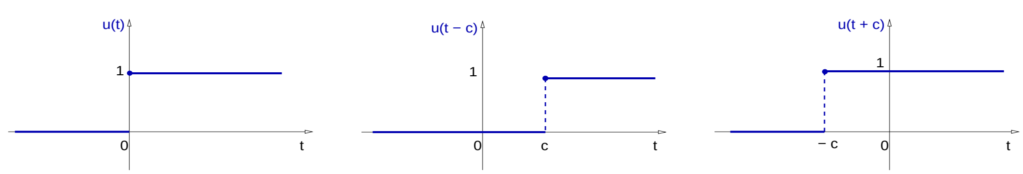
A function u is called a *step function* at $t = 0$ iff holds

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geqslant 0. \end{cases}$$

Example

Graph the step function values $u(t)$ above, and the translations $u(t - c)$ and $u(t + c)$ with $c > 0$.

Solution:

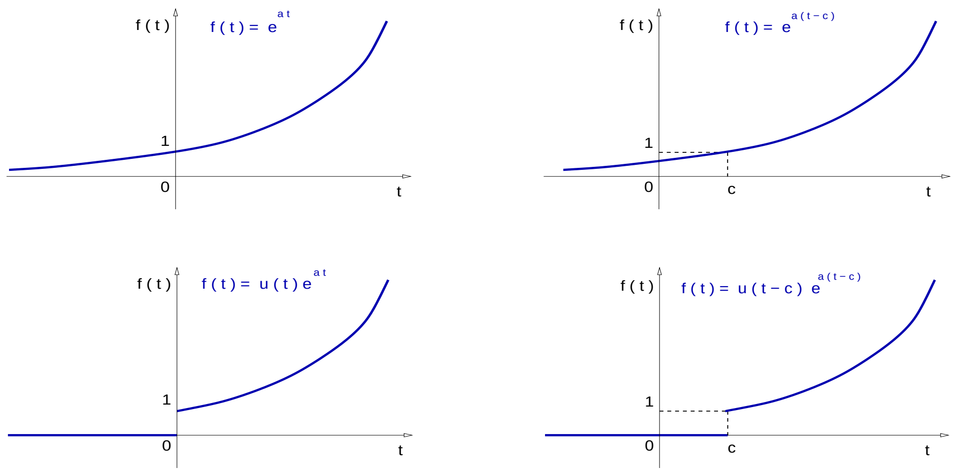


$\triangleleft$

The definition of a step function.

Remark: Given any function values $f(t)$ and $c > 0$, then $f(t-c)$ is a right translation of f and $f(t+c)$ is a left translation of f.

Example



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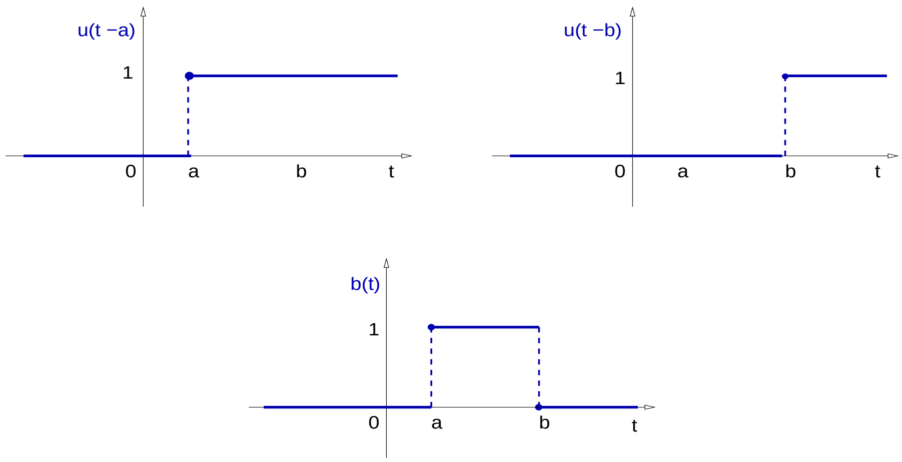
Piecewise discontinuous functions.

Example

Graph of the function $b(t) = u(t-a) - u(t-b)$, with $0 < a < b$.

Solution: The bump function b can be graphed as follows:



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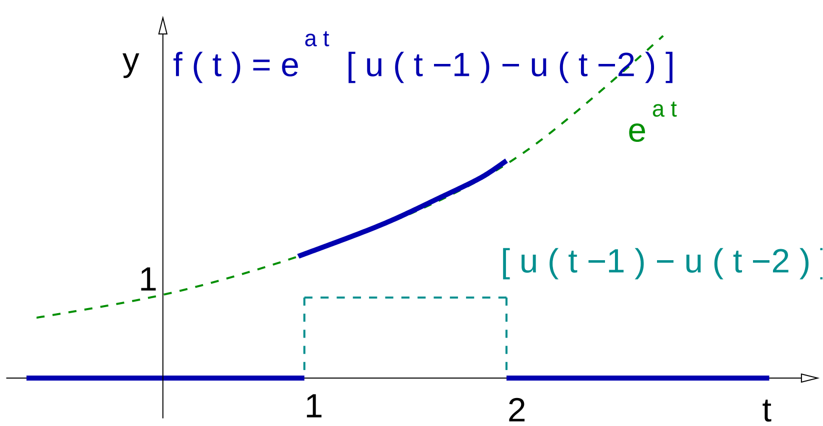
Piecewise discontinuous functions.

Example

Graph of the function $f(t) = e^{at}\left[u(t-1) - u(t-2)\right]$.

Solution:



Notation: It is common in the literature to denote the function values $u(t-c)$ as $u_c(t)$.

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The Laplace Transform of discontinuous functions.

Theorem

Given any real number $c \geqslant 0$, the following equation holds,

$$\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}, \qquad s > 0.$$

Proof:

$$\mathcal{L}[u(t-c)] = \int_0^\infty e^{-st} u(t-c)\, dt = \int_c^\infty e^{-st}\, dt,$$

$$\mathcal{L}[u(t-c)] = \lim_{N\to\infty} -\frac{1}{s}\big(e^{-Ns} - e^{-cs}\big) = \frac{e^{-cs}}{s}, \quad s > 0.$$

We conclude that $\mathcal{L}[u(t-c)] = \dfrac{e^{-cs}}{s}$. $\square$

The Laplace Transform of discontinuous functions.

Example

Compute $\mathcal{L}[3u(t-2)]$.

Solution: $\mathcal{L}[3u(t-2)] = 3\,\mathcal{L}[u(t-2)] = 3\,\dfrac{e^{-2s}}{s}$.

We conclude: $\mathcal{L}[3u(t-2)] = \dfrac{3e^{-2s}}{s}$. $\triangleleft$

Example

Compute $\mathcal{L}^{-1}\Big[\dfrac{2e^{-3s}}{s}\Big]$.

Solution: Since $\mathcal{L}[u(t-c)] = \dfrac{e^{-cs}}{s}$, for $c = 3$ we get

$\mathcal{L}^{-1}\Big[\dfrac{e^{-3s}}{s}\Big] = u(t-3)$. Therefore, $\mathcal{L}^{-1}\Big[\dfrac{2e^{-3s}}{s}\Big] = 2u(t-3)$. $\triangleleft$

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Properties of the Laplace Transform.

Theorem (Translations)

If $F(s) = \mathcal{L}[f(t)]$ *exists for* $s > a \geqslant 0$ *and* $c \geqslant 0$, *then holds*

$$\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,F(s), \qquad s > a.$$

Furthermore,

$$\mathcal{L}[e^{ct}f(t)] = F(s-c), \qquad s > a + c.$$

Remark:

- $\mathcal{L}\big[\text{translation } (uf)\big] = (\exp)\,\big(\mathcal{L}[f]\big)$.
- $\mathcal{L}\big[(\exp)\,(f)\big] = \text{translation}\big(\mathcal{L}[f]\big)$.

Equivalent notation:

- $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$,
- $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f](s-c)$.

Properties of the Laplace Transform.

Example

Compute $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big]$.

Solution: $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$, $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$.

$$\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\mathcal{L}[\sin(at)] = e^{-2s}\,\frac{a}{s^2+a^2}.$$

We conclude: $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\dfrac{a}{s^2+a^2}$. $\lhd$

Example

Compute $\mathcal{L}\big[e^{3t}\,\sin(at)\big]$.

Solution: Recall: $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f](s-c)$, $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$.

We conclude: $\mathcal{L}\big[e^{3t}\,\sin(at)\big] = \dfrac{a}{(s-3)^2+a^2}$, with $s > 3$. $\lhd$

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

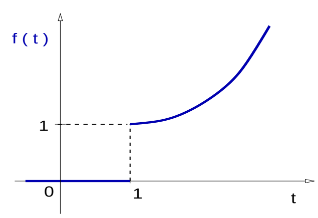
Solution: Using step function notation,

$$f(t) = u(t-1)(t^2 - 2t + 2).$$

Completing the square we obtain,

$$t^2 - 2t + 2 = (t^2 - 2t + 1) - 1 + 2 = (t-1)^2 + 1.$$

This is a parabola t^2 translated to the right by 1 and up by one. Because of the step function, this is a discontinuous function at $t = 1$.



Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Recall: $f(t) = u(t-1)\left[(t-1)^2 + 1\right]$.

This is equivalent to

$$f(t) = u(t-1)\,(t-1)^2 + u(t-1).$$

Since $\mathcal{L}[t^2] = 2/s^3$, and $\mathcal{L}[u(t-c)g(t-c)] = e^{-cs}\,\mathcal{L}[g(t)]$, then

$$\mathcal{L}[f(t)] = \mathcal{L}[u(t-1)\,(t-1)^2] + \mathcal{L}[u(t-1)] = e^{-s}\frac{2}{s^3} + e^{-s}\frac{1}{s}.$$

We conclude: $\mathcal{L}[f(t)] = \dfrac{e^{-s}}{s^3}\left(2 + s^2\right).$ $\triangleleft$

Properties of the Laplace Transform.

Remark: The inverse of the formulas in the Theorem above are:

$$\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c),$$

$$\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t).$$

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-4s}}{s^2+9}\Big]$.

Solution: $\mathcal{L}^{-1}\Big[\dfrac{e^{-4s}}{s^2+9}\Big] = \dfrac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-4s}\,\dfrac{3}{s^2+9}\Big]$.

Recall: $\mathcal{L}^{-1}\Big[\dfrac{a}{s^2+a^2}\Big] = \sin(at)$. Then, we conclude that

$$\mathcal{L}^{-1}\Big[\frac{e^{-4s}}{s^2+9}\Big] = \frac{1}{3}\,u(t-4)\,\sin\big(3(t-4)\big). \qquad \lhd$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{(s-2)}{(s-2)^2+9}\Big]$.

Solution: $\mathcal{L}^{-1}\Big[\dfrac{s}{s^2+a^2}\Big] = \cos(at)$, $\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t)$.

We conclude: $\mathcal{L}^{-1}\Big[\dfrac{(s-2)}{(s-2)^2+9}\Big] = e^{2t}\,\cos(3t)$. $\lhd$

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{2e^{-3s}}{s^2-4}\Big]$.

Solution: Recall: $\mathcal{L}^{-1}\Big[\dfrac{a}{s^2-a^2}\Big] = \sinh(at)$

and $\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c)$.

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{2e^{-3s}}{s^2-4}\Big]$.

Solution: Recall:

$$\mathcal{L}^{-1}\Big[\frac{a}{s^2-a^2}\Big] = \sinh(at), \quad \mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c).$$

$$\mathcal{L}^{-1}\Big[\frac{2e^{-3s}}{s^2-4}\Big] = \mathcal{L}^{-1}\Big[e^{-3s}\,\frac{2}{s^2-4}\Big].$$

We conclude: $\mathcal{L}^{-1}\Big[\dfrac{2e^{-3s}}{s^2-4}\Big] = u(t-3)\,\sinh\big(2(t-3)\big)$. $\lhd$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Find the roots of the denominator:

$$s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1+8}\big] \quad \Rightarrow \quad \begin{cases} s_+ = 1, \\ s_- = -2. \end{cases}$$

Therefore, $s^2+s-2 = (s-1)\,(s+2)$.

Use partial fractions to simplify the rational function:

$$\frac{1}{s^2+s-2} = \frac{1}{(s-1)\,(s+2)} = \frac{a}{(s-1)} + \frac{b}{(s+2)},$$

$$\frac{1}{s^2+s-2} = a(s+2) + b(s-1) = \frac{(a+b)\,s + (2a-b)}{(s-1)\,(s+2)}.$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Recall: $\dfrac{1}{s^2+s-2} = \dfrac{(a+b)\,s + (2a-b)}{(s-1)\,(s+2)}$

$$a+b=0, \quad 2a-b=1, \quad \Rightarrow \quad a=\frac{1}{3}, \quad b=-\frac{1}{3}.$$

$$\mathcal{L}^{-1}\Big[\frac{e^{-2s}}{s^2+s-2}\Big] = \frac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-2s}\,\frac{1}{s-1}\Big] - \frac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-2s}\,\frac{1}{s+2}\Big].$$

Recall: $\mathcal{L}^{-1}\Big[\dfrac{1}{s-a}\Big] = e^{at}$, $\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c)$,

$$\mathcal{L}^{-1}\Big[\frac{e^{-2s}}{s^2+s-2}\Big] = \frac{1}{3}\,u(t-2)\,e^{(t-2)} - \frac{1}{3}\,u(t-2)\,e^{-2(t-2)}.$$

Hence: $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big] = \dfrac{1}{3}\,u(t-2)\Big[e^{(t-2)} - e^{-2(t-2)}\Big]$. $\triangleleft$