

The Laplace Transform (Sect. 4.1).

- ▶ The definition of the Laplace Transform.
- ▶ Review: Improper integrals.
- ▶ Examples of Laplace Transforms.
- ▶ A table of Laplace Transforms.
- ▶ Properties of the Laplace Transform.
- ▶ Laplace Transform and differential equations.

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The definition of the Laplace Transform.

Definition

The function $F : D_F \rightarrow \mathbb{R}$ is the *Laplace transform* of a function $f : [0, \infty) \rightarrow \mathbb{R}$ iff for all $s \in D_F$ holds,

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt,$$

where $D_F \subset \mathbb{R}$ is the set where the integral converges.

Remark: The domain D_F of F depends on the function f .

Notation: We often denote: $F(s) = \mathcal{L}[f(t)]$.

- ▶ This notation $\mathcal{L}[\]$ emphasizes that the Laplace transform defines a map from a set of functions into a set of functions.
- ▶ Functions are denoted as $t \mapsto f(t)$.
- ▶ The Laplace transform is also a function: $f \mapsto \mathcal{L}[f]$.

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Review: Improper integrals.

Recall: Improper integrals are defined as a limit.

$$\int_{t_0}^{\infty} g(t) dt = \lim_{N \rightarrow \infty} \int_{t_0}^N g(t) dt.$$

- ▶ The integral **converges** iff the limit exists.
- ▶ The integral **diverges** iff the limit does not exist.

Example

Compute the improper integral $\int_0^{\infty} e^{-at} dt$, with $a > 0$.

Solution: $\int_0^{\infty} e^{-at} dt = \lim_{N \rightarrow \infty} \int_0^N e^{-at} dt = \lim_{N \rightarrow \infty} -\frac{1}{a} (e^{-aN} - 1).$

Since $\lim_{N \rightarrow \infty} e^{-aN} = 0$ for $a > 0$, we conclude $\int_0^{\infty} e^{-at} dt = \frac{1}{a}$. ◁

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Examples of Laplace Transforms.

Example

Compute $\mathcal{L}[1]$.

Solution: We have to find the Laplace Transform of $f(t) = 1$. Following the definition we obtain,

$$\mathcal{L}[1] = \int_0^{\infty} e^{-st} 1 dt = \int_0^{\infty} e^{-st} dt$$

But $\int_0^{\infty} e^{-at} dt = \frac{1}{a}$ for $a > 0$, and diverges for $a \leq 0$.

Therefore $\mathcal{L}[1] = \frac{1}{s}$, for $s > 0$, and $\mathcal{L}[1]$ does not exist for $s \leq 0$.

In other words, $F(s) = \mathcal{L}[1]$ is the function $F : D_F \rightarrow \mathbb{R}$ given by

$$f(t) = 1, \quad F(s) = \frac{1}{s}, \quad D_F = (0, \infty). \quad \triangleleft$$

Examples of Laplace Transforms.

Example

Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.

Solution: Following the definition of Laplace Transform,

$$\mathcal{L}[e^{at}] = \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt.$$

We have seen that the improper integral is given by

$$\int_0^{\infty} e^{-(s-a)t} dt = \frac{1}{(s-a)} \quad \text{for } (s-a) > 0.$$

We conclude that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s > a$. In other words,

$$f(t) = e^{at}, \quad F(s) = \frac{1}{(s-a)}, \quad s > a. \quad \triangleleft$$

Examples of Laplace Transforms.

Example

Compute $\mathcal{L}[\sin(at)]$, where $a \in \mathbb{R}$.

Solution: In this case we need to compute

$$\mathcal{L}[\sin(at)] = \lim_{N \rightarrow \infty} \int_0^N e^{-st} \sin(at) dt.$$

Integrating by parts twice it is not difficult to obtain:

$$\begin{aligned} & \int_0^N e^{-st} \sin(at) dt = \\ & -\frac{1}{s} [e^{-st} \sin(at)] \Big|_0^N - \frac{a}{s^2} [e^{-st} \cos(at)] \Big|_0^N - \frac{a^2}{s^2} \int_0^N e^{-st} \sin(at) dt. \end{aligned}$$

This identity implies

$$\left(1 + \frac{a^2}{s^2}\right) \int_0^N e^{-st} \sin(at) dt = -\frac{1}{s} [e^{-st} \sin(at)] \Big|_0^N - \frac{a}{s^2} [e^{-st} \cos(at)] \Big|_0^N.$$

Examples of Laplace Transforms.

Example

Compute $\mathcal{L}[\sin(at)]$, where $a \in \mathbb{R}$.

Solution: Recall the identity:

$$\left(1 + \frac{a^2}{s^2}\right) \int_0^N e^{-st} \sin(at) dt = -\frac{1}{s} [e^{-st} \sin(at)] \Big|_0^N - \frac{a}{s^2} [e^{-st} \cos(at)] \Big|_0^N.$$

Hence, it is not difficult to see that

$$\left(\frac{s^2 + a^2}{s^2}\right) \int_0^\infty e^{-st} \sin(at) dt = \frac{a}{s^2}, \quad s > 0,$$

which is equivalent to

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}, \quad s > 0. \quad \triangleleft$$

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A table of Laplace Transforms.

$f(t) = 1$	$F(s) = \frac{1}{s}$	$s > 0,$
$f(t) = e^{at}$	$F(s) = \frac{1}{s - a}$	$s > \max\{a, 0\},$
$f(t) = t^n$	$F(s) = \frac{n!}{s^{(n+1)}}$	$s > 0,$
$f(t) = \sin(at)$	$F(s) = \frac{a}{s^2 + a^2}$	$s > 0,$
$f(t) = \cos(at)$	$F(s) = \frac{s}{s^2 + a^2}$	$s > 0,$
$f(t) = \sinh(at)$	$F(s) = \frac{a}{s^2 - a^2}$	$s > a ,$
$f(t) = \cosh(at)$	$F(s) = \frac{s}{s^2 - a^2}$	$s > a ,$
$f(t) = t^n e^{at}$	$F(s) = \frac{n!}{(s - a)^{(n+1)}}$	$s > \max\{a, 0\},$
$f(t) = e^{at} \sin(bt)$	$F(s) = \frac{b}{(s - a)^2 + b^2}$	$s > \max\{a, 0\}.$

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Properties of the Laplace Transform.

Theorem (Sufficient conditions)

If the function $f : [0, \infty) \rightarrow \mathbb{R}$ is piecewise continuous and there exist positive constants k and a such that

$$|f(t)| \leq k e^{at},$$

then the Laplace Transform of f exists for all $s > a$.

Theorem (Linear combination)

If the $\mathcal{L}[f]$ and $\mathcal{L}[g]$ are well-defined and a, b are constants, then

$$\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g].$$

Proof: Integration is a linear operation:

$$\int [af(t) + bg(t)] dt = a \int f(t) dt + b \int g(t) dt.$$

Properties of the Laplace Transform.

Theorem (Derivatives)

If the $\mathcal{L}[f]$ and $\mathcal{L}[f']$ are well-defined, then holds,

$$\mathcal{L}[f'] = s \mathcal{L}[f] - f(0). \quad (1)$$

Furthermore, if $\mathcal{L}[f'']$ is well-defined, then it also holds

$$\mathcal{L}[f''] = s^2 \mathcal{L}[f] - s f(0) - f'(0). \quad (2)$$

Proof of Eq (2): Use Eq. (1) twice:

$$\mathcal{L}[f''] = \mathcal{L}[(f')'] = s \mathcal{L}[(f')] - f'(0) = s(s \mathcal{L}[f] - f(0)) - f'(0),$$

that is,

$$\mathcal{L}[f''] = s^2 \mathcal{L}[f] - s f(0) - f'(0).$$

Properties of the Laplace Transform.

Proof of Eq (1): Recall the definition of the Laplace Transform,

$$\mathcal{L}[f'] = \int_0^{\infty} e^{-st} f'(t) dt = \lim_{n \rightarrow \infty} \int_0^n e^{-st} f'(t) dt$$

Integrating by parts,

$$\lim_{n \rightarrow \infty} \int_0^n e^{-st} f'(t) dt = \lim_{n \rightarrow \infty} \left[\left(e^{-st} f(t) \right) \Big|_0^n - \int_0^n (-s) e^{-st} f(t) dt \right]$$

$$\mathcal{L}[f'] = \lim_{n \rightarrow \infty} \left[e^{-sn} f(n) - f(0) \right] + s \int_0^{\infty} e^{-st} f(t) dt = -f(0) + s \mathcal{L}[f],$$

where we used that $\lim_{n \rightarrow \infty} e^{-sn} f(n) = 0$ for s big enough, and we also used that $\mathcal{L}[f]$ is well-defined.

We then conclude that $\mathcal{L}[f'] = s \mathcal{L}[f] - f(0)$.

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Laplace Transform and differential equations.

Remark: Laplace Transforms can be used to find solutions to differential equations with **constant coefficients**.

Idea of the method:

$$\mathcal{L} \left[\begin{array}{l} \text{Differential Eq.} \\ \text{for } y(t). \end{array} \right] \xrightarrow{(1)} \text{Algebraic Eq.} \xrightarrow{(2)} \text{for } \mathcal{L}[y(t)].$$

$$\xrightarrow{(2)} \begin{array}{l} \text{Solve the} \\ \text{Algebraic Eq.} \\ \text{for } \mathcal{L}[y(t)]. \end{array} \xrightarrow{(3)} \begin{array}{l} \text{Transform back} \\ \text{to obtain } y(t). \\ \text{(Using the table.)} \end{array}$$

Laplace Transform and differential equations.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \quad y(0) = 3.$$

Solution: We know the solution: $y(t) = 3e^{-2t}$.

(1): Compute the Laplace transform of the differential equation,

$$\mathcal{L}[y' + 2y] = \mathcal{L}[0] \Rightarrow \mathcal{L}[y' + 2y] = 0.$$

Find an algebraic equation for $\mathcal{L}[y]$. Recall linearity:

$$\mathcal{L}[y'] + 2\mathcal{L}[y] = 0.$$

Also recall the property: $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$, that is,

$$\left[s\mathcal{L}[y] - y(0) \right] + 2\mathcal{L}[y] = 0 \Rightarrow (s + 2)\mathcal{L}[y] = y(0).$$

Laplace Transform and differential equations.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \quad y(0) = 3.$$

Solution: Recall: $(s + 2)\mathcal{L}[y] = y(0)$.

(2): Solve the algebraic equation for $\mathcal{L}[y]$.

$$\mathcal{L}[y] = \frac{y(0)}{s + 2}, \quad y(0) = 3, \Rightarrow \mathcal{L}[y] = \frac{3}{s + 2}.$$

(3): Transform back to $y(t)$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s - a} \Rightarrow \frac{3}{s + 2} = 3\mathcal{L}[e^{-2t}] \Rightarrow \frac{3}{s + 2} = \mathcal{L}[3e^{-2t}].$$

Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}] \Rightarrow y(t) = 3e^{-2t}$.

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