The Euler equation (Sect. 3.2).

- We study the Euler Equation:
  $(x - x_0)^2\, y'' + p_0\,(x - x_0)\, y' + q_0\, y = 0.$
- Solutions to the Euler equation near x_0.
- The roots of the indicial polynomial.
  - Different real roots.
  - Repeated roots.
  - Different complex roots.

The Euler equation

Definition

Given real constants p_0, q_0, the *Euler differential equation* for the unknown y with singular point at $x_0 \in R$ is given by

$$(x - x_0)^2\, y'' + p_0\,(x - x_0)\, y' + q_0\, y = 0.$$

Remarks:

- The Euler equation has variable coefficients.
- Functions $y(x) = e^{rx}$ are not solutions of the Euler equation.
- The point $x_0 \in \mathbb{R}$ is a singular point of the equation.
- The particular case $x_0 = 0$ is is given by

$$x^2\, y'' + p_0\, x\, y' + q_0\, y = 0.$$

The Euler equation (Sect. 3.2).

- We study the Euler Equation:
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Solutions to the Euler equation near x_0.

Summary of the main idea:

- The main idea to find solution to the constant coefficients equation $y'' + a_1\, y' + a_0\, y = 0$ was to look for functions of the form $y(x) = e^{rx}$. The exponential cancels out from the equation and we obtain an equation only for r without x,

$$\left(r^2 + a_1\, r + a_0\right) e^{rx} = 0 \quad \Leftrightarrow \quad \left(r^2 + a_1\, r + a_0\right) = 0. \qquad (1)$$

- In the case of the Euler equation $x^2\, y'' + p_0\, x\, y' + q_0\, y = 0$ the exponential functions e^{rx} do not have the property given in Eq. (1), since

$$\left(x^2\, r^2 + p_0\, x\, r + q_0\right) e^{rx} = 0 \quad \Leftrightarrow \quad x^2\, r^2 + p_0\, x\, r + q_0 = 0,$$

  but the later equation still involves the variable x.

Solutions to the Euler equation near x_0.

Summary of the main idea: Look for solutions like $y(x) = x^r$.

These function have the following property:

$$y'(x) = r\,x^{r-1} \quad \Rightarrow \quad x\,y'(x) = r\,x^r;$$

$$y''(x) = r(r-1)\,x^{r-2} \quad \Rightarrow \quad x^2\,y''(x) = r(r-1)\,x^r.$$

Introduce $y = x^r$ into Euler's equation $x^2\,y'' + p_0\,x\,y' + q_0\,y = 0$, for $x \neq 0$ we obtain

$$\big[r(r-1) + p_0 r + q_0\big]\,x^r = 0 \quad \Leftrightarrow \quad r(r-1) + p_0 r + q_0 = 0.$$

The last equation involves only r, not x.

This equation is called the indicial equation, and is also called the Euler characteristic equation.

Solutions to the Euler equation near x_0.

Theorem (Euler equation, $x_0 = 0$)

Given p_0, q_0, $x_0 \in \mathbb{R}$, consider the Euler equation

$$x^2\,y'' + p_0\,x\,y' + q_0\,y = 0. \tag{2}$$

Let r_+, r_- be solutions of $r(r-1) + p_0 r + q_0 = 0$.

(a) *If $r_+ \neq r_-$, then a general solution of Eq. (2) is*

$$y(x) = c_0|x|^{r_+} + c_1|x|^{r_-}, \quad x \neq 0, \quad c_0,\ c_1 \in \mathbb{R}\ (\text{or } \mathbb{C}).$$

(b) *If $r_+ = r_- = \hat{r}$, then a real-valued general solution of Eq. (2) is*

$$y(x) = \big[c_0 + c_1 \ln|x|\big]\,|x|^{\hat{r}}, \quad x \neq 0, \quad c_0,\ c_1 \in \mathbb{R}.$$

Given $x_1 \neq 0$, y_0, $y_1 \in \mathbb{R}$, there is a unique solution to the IVP

$$x^2\,y'' + p_0\,x\,y' + q_0\,y = 0, \quad y(x_1) = y_0, \quad y'(x_1) = y_1.$$

Solutions to the Euler equation near x_0.

Theorem (Euler equation, $x_0 \neq 0$)

Given p_0, q_0, $x_0 \in \mathbb{R}$, consider the Euler equation

$$(x - x_0)^2\, y'' + p_0\,(x - x_0)\, y' + q_0\, y = 0. \tag{3}$$

Let r_+, r_- be solutions of $r(r-1) + p_0 r + q_0 = 0$.

(a) *If $r_+ \neq r_-$, then a general solution of Eq. (3) is*

$$y(x) = c_0 |x - x_0|^{r_+} + c_1 |x - x_0|^{r_-}, \quad x \neq x_0, \quad c_0,\ c_1 \in \mathbb{R}\ (\text{or } \mathbb{C}).$$

(b) *If $r_+ = r_- = \hat{r}$, then a real-valued general solution of Eq. (3) is*

$$y(x) = \Big[c_0 + c_1 \ln|x - x_0|\Big]\, |x - x_0|^{\hat{r}}, \quad x \neq x_0, \quad c_0,\ c_1 \in \mathbb{R}.$$

Given $x_1 \neq x_0$, y_0, $y_1 \in \mathbb{R}$, there is a unique solution to the IVP

$$(x - x_0)^2\, y'' + p_0\,(x - x_0)\, y' + q_0\, y = 0, \quad y(x_1) = y_0, \quad y'(x_1) = y_1.$$

The Euler equation (Sect. 3.2).

- We study the Euler Equation: $(x - x_0)^2\, y'' + p_0\,(x - x_0)\, y' + q_0\, y = 0$.
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  - **Different real roots.**
  - Repeated roots.
  - Different complex roots.

Different real roots.

Example

Find the general solution of the Euler equation

$$x^2\, y'' + 4x\, y' + 2\, y = 0.$$

Solution: We look for solutions of the form $y(x) = x^r$,

$$x\, y'(x) = r x^r, \qquad x^2\, y''(x) = r(r-1)\, x^r.$$

Introduce $y(x) = x^r$ into Euler equation,

$$\big[r(r-1) + 4r + 2\big]\, x^r = 0 \quad \Leftrightarrow \quad r(r-1) + 4r + 2 = 0.$$

The solutions of $r^2 + 3r + 2 = 0$ are given by

$$r_\pm = \frac{1}{2}\big[-3 \pm \sqrt{9-8}\big] \quad \Rightarrow \quad r_+ = -1 \qquad r_- = -2.$$

The general solution is $y(x) = c_1\, |x|^{-1} + c_2\, |x|^{-2}$. ◁

The Euler equation (Sect. 3.2).

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Repeated roots.

Example

Find the general solution of $x^2 y'' - 3x\, y' + 4\, y = 0.$

Solution: We look for solutions of the form $y(x) = x^r$,

$$x\, y'(x) = rx^r, \qquad x^2\, y''(x) = r(r-1)\, x^r.$$

Introduce $y(x) = x^r$ into Euler equation,

$$\big[r(r-1) - 3r + 4\big]\, x^r = 0 \quad \Leftrightarrow \quad r(r-1) - 3r + 4 = 0.$$

The solutions of $r^2 - 4r + 4 = 0$ are given by

$$r_{\pm} = \frac{1}{2}\big[4 \pm \sqrt{16 - 16}\big] \quad \Rightarrow \quad r_+ = r_- = 2.$$

Two linearly independent solutions are

$$y_1(x) = x^2, \qquad y_2 = x^2 \ln(|x|).$$

The general solution is $y(x) = c_1\, x^2 + c_2\, x^2 \ln(|x|).$ $\triangleleft$

The Euler equation (Sect. 3.2).

- We study the Euler Equation: $(x - x_0)^2\, y'' + p_0\,(x - x_0)\, y' + q_0\, y = 0.$
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  - **Different complex roots.**

Different complex roots.

Example

Find the general solution of the Euler equation

$$x^2\, y'' - 3x\, y' + 13\, y = 0.$$

Solution: We look for solutions of the form $y(x) = x^r$,

$$x\, y'(x) = r x^r, \qquad x^2\, y''(x) = r(r-1)\, x^r.$$

Introduce $y(x) = x^r$ into Euler equation

$$\big[r(r-1) - 3r + 13\big]\, x^r = 0 \quad \Leftrightarrow \quad r(r-1) - 3r + 13 = 0.$$

The solutions of the indicial equation $r^2 - 4r + 13 = 0$ are

$$r_\pm = \frac{1}{2}\big[4 \pm \sqrt{16 - 52}\big] \;\Rightarrow\; r_\pm = \frac{1}{2}\big[4 \pm \sqrt{-36}\big] \;\Rightarrow\; \begin{cases} r_+ = 2 + 3i \\ r_- = 2 - 3i. \end{cases}$$

The general solution is $y(x) = c_1\, |x|^{(2+3i)} + c_2\, |x|^{(2-3i)}$. ◁

Different complex roots.

Theorem (Real-valued fundamental solutions)

If $p_0, q_0 \in \mathbb{R}$ satisfy that $\big[(p_0 - 1)^2 - 4q_0\big] < 0$, then the indicial polynomial $p(r) = r(r-1) + p_0 r + q_0$ of the Euler equation

$$x^2\, y'' + p_0 x\, y' + q_0\, y = 0 \tag{4}$$

has complex roots $r_+ = \alpha + i\beta$ and $r_- = \alpha - i\beta$, where

$$\alpha = -\frac{(p_0 - 1)}{2}, \qquad \beta = \frac{1}{2}\sqrt{4q_0 - (p_0 - 1)^2}.$$

A complex-valued fundamental set of solution to Eq. (4) is

$$\tilde{y}_1(x) = |x|^{(\alpha + i\beta)}, \qquad \tilde{y}_2(x) = |x|^{(\alpha - i\beta)}.$$

A real-valued fundamental set of solutions to Eq. (4) is

$$y_1(x) = |x|^\alpha \cos\big(\beta \ln|x|\big), \qquad y_2(x) = |x|^\alpha \sin\big(\beta \ln|x|\big).$$

Different complex roots.

Proof: Given $\tilde{y}_1 = |x|^{(\alpha+i\beta)}$ and $\tilde{y}_2 = |x|^{(\alpha-i\beta)}$, introduce

$$y_1 = \frac{1}{2}\big(\tilde{y}_1 + \tilde{y}_2\big), \qquad y_2 = \frac{1}{2i}\big(\tilde{y}_1 - \tilde{y}_2\big).$$

Use another Euler equation to rewrite $\tilde{y}_1$ and $\tilde{y}_2$,

$$\tilde{y}_1 = |x|^{(\alpha+i\beta)} = |x|^{\alpha}\,|x|^{i\beta} = |x|^{\alpha}\,e^{\ln(|x|^{i\beta})} = |x|^{\alpha}\,e^{i\beta\ln(|x|)}.$$

$$\tilde{y}_1 = |x|^{\alpha}\big[\cos\big(\beta\ln|x|\big) + i\sin\big(\beta\ln|x|\big)\big],$$

$$\tilde{y}_2 = |x|^{\alpha}\big[\cos\big(\beta\ln|x|\big) - i\sin\big(\beta\ln|x|\big)\big].$$

We conclude that

$$y_1(x) = |x|^{\alpha}\cos\big(\beta\,\ln|x|\big), \qquad y_2(x) = |x|^{\alpha}\sin\big(\beta\,\ln|x|\big).$$

□

Different complex roots.

Example

Find a real-valued general solution of the Euler equation

$$x^2\,y'' - 3x\,y' + 13\,y = 0.$$

Solution: The indicial equation is $r(r-1) - 3r + 13 = 0$.

The solutions of the indicial equations are

$$r^2 - 4r + 13 = 0 \quad\Rightarrow\quad r_+ = 2 + 3i, \quad r_- = 2 - 3i.$$

A complex-valued general solution is

$$y(x) = \tilde{c}_1\,|x|^{(2+3i)} + \tilde{c}_2\,|x|^{(2-3i)} \quad \tilde{c}_1,\ \tilde{c}_2 \in \mathbb{C}.$$

A real-valued general solution is

$$y(x) = c_1\,|x|^2\cos\big(3\ln|x|\big) + c_2\,|x|^2\sin\big(3\ln|x|\big), \quad c_1,\ c_2 \in \mathbb{R}. \quad \triangleleft$$