Second order linear homogeneous ODE (Sect. 2.4).

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Repeated roots as a limit case.
- Main result for repeated roots.
- Reduction of the order method:
  - Constant coefficients equations.
  - Variable coefficients equations.

Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.

Summary:
Given constants a_1, $a_0 \in \mathbb{R}$, consider the differential equation

$$y'' + a_1 y' + a_0 y = 0$$

with characteristic polynomial having roots

$$r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0}.$$

(1) If $a_1^2 - 4a_0 > 0$, then $y_1(t) = e^{r_+ t}$ and $y_2(t) = e^{r_- t}$.

(2) If $a_1^2 - 4a_0 < 0$, then introducing $\alpha = -\frac{a_1}{2}$, $\beta = \frac{1}{2}\sqrt{4a_0 - a_1^2}$,

$$y_1(t) = e^{\alpha t}\cos(\beta t), \qquad y_2(t) = e^{\alpha t}\sin(\beta t).$$

(3) If $a_1^2 - 4a_0 = 0$, then $y_1(t) = e^{-\frac{a_1}{2}t}$.

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Question:

Consider the case (3), with $a_1^2 - 4a_0 = 0$, that is, $a_0 = \dfrac{a_1^2}{4}$.

- Does the equation

$$y'' + a_1 y' + \frac{a_1^2}{4} y = 0$$

have two linearly independent solutions?

- Or, is every solution to the equation above proportional to

$$y_1(t) = e^{-\frac{a_1}{2} t} \qquad ?$$

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Repeated roots as a limit case.

Remark:

- Case (3), where $4a_0 - a_1^2 = 0$ can be obtained as the limit $\beta \to 0$ in case (2).
- Let us study the solutions of the differential equation in the case (2) as $\beta \to 0$ for fixed t.
- Since $\cos(\beta t) \to 1$ as $\beta \to 0$, we conclude that

$$y_{1\beta}(t) = e^{-\frac{a_1}{2}\,t} \cos(\beta t) \to e^{-\frac{a_1}{2}\,t} = y_1(t).$$

- Since $\dfrac{\sin(\beta t)}{\beta t} \to 1$ as $\beta \to 0$, that is, $\sin(\beta t) \to \beta t$,

$$y_{2\beta}(t) = e^{-\frac{a_1}{2}\,t} \sin(\beta t) \to \beta t\, e^{-\frac{a_1}{2}\,t} \to 0.$$

- Is $y_2(t) = t\, y_1(t)$ solution of the differential equation? Introducing y_2 in the differential equation one obtains: Yes.
- Since y_2 is not proportional to y_1, the functions y_1, y_2 are a fundamental set for the differential equation in case (3).

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Main result for repeated roots.

Theorem

If a_1, $a_0 \in R$ satisfy that $a_1^2 = 4a_0$, then the functions

$$y_1(t) = e^{-\frac{a_1}{2}\,t}, \qquad y_2(t) = t\,e^{-\frac{a_1}{2}\,t},$$

are a fundamental solution set for the differential equation

$$y'' + a_1 y' + a_0 y = 0.$$

Example

Find the general solution of $9y'' + 6y' + y = 0$.

Solution: The characteristic equation is $9r^2 + 6r + 1 = 0$, so

$$r_\pm = \frac{1}{(2)(9)}\left[-6 \pm \sqrt{36-36}\right] \quad \Rightarrow \quad r_\pm = -\frac{1}{3}.$$

The Theorem above implies that the general solution is

$$y(t) = c_1\,e^{-t/3} + c_2\,te^{-t/3}.$$

$\triangleleft$

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Reduction of the order method: Constant coefficients.

Proof case $a_1^2 - 4a_0 = 0$:
Recall: The characteristic equation is $r^2 + a_1 r + a_0 = 0$, and its solutions are $r_\pm = (1/2)\big[-a_1 \pm \sqrt{a_1^2 - 4a_0}\big]$.

The hypothesis $a_1^2 = 4a_0$ implies $r_+ = r_- = -a_1/2$.

So, the solution r_+ of the characteristic equation satisfies both

$$r_+^2 + a_1 r_+ + a_0 = 0, \qquad 2r_+ + a_1 = 0.$$

It is clear that $y_1(t) = e^{r_+ t}$ is solutions of the differential equation.

A second solution y_2 not proportional to y_1 can be found as follows: (D'Alembert $\sim$ 1750.)

Express: $y_2(t) = v(t)\, y_1(t)$, and find the equation that function v satisfies from the condition $y_2'' + a_1 y_2' + a_0 y_2 = 0$.

Reduction of the order method: Constant coefficients.

Recall: $y_2 = v y_1$ and $y_2'' + a_1 y_2' + a_0 y_2 = 0$. So, $y_2 = v e^{r_+ t}$ and

$$y_2' = v' e^{r_+ t} + r_+ v e^{r_+ t}, \qquad y_2'' = v'' e^{r_+ t} + 2r_+ v' e^{r_+ t} + r_+^2 v e^{r_+ t}.$$

Introducing this information into the differential equation

$$\big[v'' + 2r_+ v' + r_+^2 v\big]\, e^{r_+ t} + a_1 \big[v' + r_+ v\big]\, e^{r_+ + t} + a_0 v\, e^{r_+ t} = 0.$$

$$\big[v'' + 2r_+ v' + r_+^2 v\big] + a_1 \big[v' + r_+ v\big] + a_0 v = 0$$

$$v'' + \big(2r_+ + a_1\big)\, v' + \big(r_+^2 + a_1 r_+ + a_0\big)\, v = 0$$

Recall that r_+ satisfies: $r_+^2 + a_1 r_+ + a_0 = 0$ and $2r_+ + a_1 = 0$.

$$v'' = 0 \quad \Rightarrow \quad v = (c_1 + c_2 t) \quad \Rightarrow \quad y_2 = (c_1 + c_2 t)\, e^{r_+ t}.$$

Reduction of the order method: Constant coefficients.

Recall: We have obtained that $y_2(t) = (c_1 + c_2 t)\, e^{r_+ t}$.

If $c_2 = 0$, then $y_2 = c_1 e^{r_+ t}$ and $y_1 = e^{r_+ t}$ are linearly dependent functions.

If $c_2 \neq 0$, then $y_2 = (c_1 + c_2 t)\, e^{r_+ t}$ and $y_1 = e^{r_+ t}$ are linearly independent functions.

Simplest choice: $c_1 = 0$ and $c_2 = 1$. Then, a fundamental solution set to the differential equation is

$$y_1(t) = e^{r_+ t}, \qquad y_2(t) = t\, e^{r_+ t} \qquad \square$$

The general solution to the differential equation is

$$y(t) = \tilde{c}_1\, e^{r_+ t} + \tilde{c}_2\, t e^{r_+ t}.$$

Reduction of the order method: Constant coefficients.

Example

Find the solution to the initial value problem

$$9y'' + 6y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = \frac{5}{3}.$$

Solution: The solutions of $9r^2 + 6r + 1 = 0$, are $r_+ = r_- = -\dfrac{1}{3}$.

The Theorem above says that the general solution is

$$y(t) = c_1 e^{-t/3} + c_2 t e^{-t/3} \;\Rightarrow\; y'(t) = -\frac{c_1}{3} e^{-t/3} + c_2 \left(1 - \frac{t}{3}\right) e^{-t/3}.$$

The initial conditions imply that

$$\left.\begin{aligned} 1 &= y(0) = c_1, \\ \frac{5}{3} &= y'(0) = -\frac{c_1}{3} + c_2 \end{aligned}\right\} \quad \Rightarrow \quad c_1 = 1, \qquad c_2 = 2.$$

We conclude that $y(t) = (1 + 2t)\, e^{-t/3}$. $\triangleleft$

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Reduction of the order method: Variable coefficients.

Remark: The same idea used to prove the constant coefficients Theorem above can be used in variable coefficients equations.

Theorem
Given continuous functions p, $q : (t_1, t_2) \to \mathbb{R}$, *let* $y_1 : (t_1, t_2) \to \mathbb{R}$ *be a solution of*

$$y'' + p(t)\, y' + q(t)\, y = 0,$$

If the function $v : (t_1, t_2) \to \mathbb{R}$ *is solution of*

$$y_1(t)\, v'' + \big[2y'(t) + p(t)y_1(t)\big]\, v' = 0. \qquad (1)$$

then the functions y_1 *and* $y_2 = v\, y_1$ *are fundamental solutions to the differential equation above.*

Remark: The reason for the name Reduction of order method is that the function v does not appear in Eq. (1). This is a first order equation in v'.

Reduction of the order method: Variable coefficients.

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Express $y_2(t) = v(t)\, y_1(t)$. The equation for v comes from $t^2 y_2'' + 2t y_2' - 2y_2 = 0$. We need to compute

$$y_2 = v\, t, \qquad y_2' = t\, v' + v, \qquad y_2'' = t\, v'' + 2v'.$$

So, the equation for v is given by

$$t^2 \big(t\, v'' + 2v'\big) + 2t\big(t\, v' + v\big) - 2t\, v = 0$$

$$t^3\, v'' + (2t^2 + 2t^2)\, v' + (2t - 2t)\, v = 0$$

$$t^3\, v'' + (4t^2)\, v' = 0 \quad \Rightarrow \quad v'' + \frac{4}{t} v' = 0.$$

Reduction of the order method: Variable coefficients.

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Recall: $v'' + \frac{4}{t} v' = 0.$

This is a first order equation for $w = v'$, given by $w' + \frac{4}{t} w = 0$, so

$$\frac{w'}{w} = -\frac{4}{t} \;\Rightarrow\; \ln(w) = -4\ln(t) + c_0 \;\Rightarrow\; w(t) = c_1 t^{-4}, \; c_1 \in \mathbb{R}.$$

Integrating w we obtain v, that is, $v = c_2 t^{-3} + c_3$, with $c_2, c_3 \in \mathbb{R}$. Recalling that $y_2 = t\, v$ we then conclude that $y_2 = c_2 t^{-2} + c_3 t$. Choosing $c_2 = 1$ and $c_3 = 0$ we obtain the fundamental solutions $y_1(t) = t$ and $y_2(t) = \frac{1}{t^2}$. $\lhd$

Reduction of the order method: Variable coefficients.

Proof of the Theorem: The choice of $y_2 = v y_1$ implies

$$y_2' = v'\, y_1 + v\, y_1', \qquad y_2'' = v''\, y_1 + 2v'\, y_1' + v\, y_1''.$$

This information introduced into the differential equation says that

$$(v''\, y_1 + 2v'\, y_1' + v\, y_1'') + p\,(v'\, y_1 + v\, y_1') + qv\, y_1 = 0$$

$$y_1\, v'' + (2y_1' + p\, y_1)\, v' + (y_1'' + p\, y_1' + q\, y_1)\, v = 0.$$

The function y_1 is solution of $y_1'' + p\, y_1' + q\, y_1 = 0$.

Then, the equation for v is given by Eq. (1), that is,

$$y_1\, v'' + (2y_1' + p\, y_1)\, v' = 0.$$

Reduction of the order method: Variable coefficients.

Proof: Recall $y_1\, v'' + (2y_1' + p\, y_1)\, v' = 0$. We now need to show that y_1 and $y_2 = v y_1$ are linearly independent.

$$W_{y_1 y_2} = \begin{vmatrix} y_1 & v y_1 \\ y_1' & (v' y_1 + v y_1') \end{vmatrix} = y_1(v' y_1 + v y_1') - v y_1 y_1'.$$

We obtain $W_{y_1 y_2} = v' y_1^2$. We need to find v'. Denote $w = v'$, so

$$y_1\, w' + (2y_1' + p\, y_1)\, w = 0 \quad \Rightarrow \quad \frac{w'}{w} = -2\frac{y_1'}{y_1} - p.$$

Let P be a primitive of p, that is, $P'(t) = p(t)$, then

$$\ln(w) = -2\ln(y_1) - P \;\Rightarrow\; w = e^{[\ln(y_1^{-2}) - P]} \;\Rightarrow\; w = y_1^{-2}\, e^{-P}.$$

We obtain $v' y_1^2 = e^{-P}$, hence $W_{y_1 y_2} = e^{-P}$, which is non-zero.
We conclude that y_1 and $y_2 = v y_1$ are linearly independent. $\square$